OBJECTIVE TESTS AND MATHEMATICAL LEARNING

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OLIVER AND BOYD

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CHAPTER 1

WHAT ARE OBJECTIVE TESTS?

Objective tests are tests composed of questions to each of which there is a unique acceptable answer. Because there is only one acceptable answer, such tests can be marked by any intelligent person provided with an answer key. In this sense they are unlike the extended-answer or 'essay' type of questions, which require markers who are skilled in the topic being tested. In such extended-answer tests, the subjective opinions of examiners regarding the mark to be given often differ, and this introduces a source of error into the total mark obtained in the test.

It should be realized, however, that marks on objective tests are not free of error just because they are objectively marked. The selection of questions, the wording of questions, and the emphasis given to different topics and skills are all subjective decisions made by the examiner in the devising of the test, and so are potential sources of 'error' in regard to any final score on the test. So objective tests are subjectively structured, written, and designed. They are objectively marked.

In objective tests, the answers may be presented to the student in *multiple-choice* form,[1] in which case the student has to choose the correct answer or set of answers from a number of alternatives. A very simple form is the true-false answer. This type of presentation facilitates fast machine scoring, and has advantages when the choices incorporate shades of meaning which reflect fine distinctions in understanding.[2] In as much as guessing, more or less inspired, occurs with this form of test, there is some added uncertainty (unreliability) in the final score. Where the number of questions is large, however, this source of unreliability is considered relatively insignificant when compared to other possible sources of error. And guessing also applies, of course, in extended-answer

[1] These superscripts refer to the numbers of the test items presented at the end of the chapter.

questions. How often is a student certain that what he writes is correct?

In mathematics, where the answers are often numerical or symbolic in form, many questions may be left *open-ended*,[3] and students are required to write in the unequivocal answer.

Apart from the ease and objectivity of marking objective tests, their main attraction is their sampling efficiency. In quite a short time a comparatively large number of questions, covering different content areas and mathematical skills, may be asked. At Form 4 or Form 6 level about forty quite complex questions per hour can be answered. As we shall see later (Chapter 7) this adequacy of sampling is important if the results of students on the test are to have much meaning.

There are advantages, then, to be gained through the use of objective tests. They may be used to give relatively accurate and reliable measures of many mathematical skills. But they may not be used to measure a student's ability to synthesize a proof, to present a mathematical argument or to give direct evidence of imaginative, divergent thinking. They may, however, test diverse skills, simple,[4] complex,[5] naive[6] or sophisticated,[7] involving simple recall[8] or advanced analytical[9] or intuitive ingenuity.[10]

The objective tests to be discussed in this book are concerned only with cognitive behaviours. Some legitimate aims of mathematics teaching are concerned with attitudes. Many teachers claim: 'We want to interest the children, to develop in them a love of mathematics, an appreciation of its power and elegance.' Whilst the aim may be applauded, objective tests of the type to be discussed here are inappropriate for the measurement of attitudes.*

*Anyway, perhaps such tests should be taken only by the teacher, for it has always seemed to me that teachers who complain that students lack interest and appreciation of mathematics are rather like comedians who complain that their audiences have no sense of humour. In fact, they need a new technique, or new jokes.

EXAMPLES OF TEST ITEM TYPES IN CHAPTER 1

1 Multiple-Choice Form

If $S = \{1, 2, 3\}$ and $T = \{3, 4, 5\}$, then which of the following is a true statement?

A $S \cup T = \{3\}$ **C** $S \cup T = \{T\}$

B $S \cap T = \{S\}$ **D** $S \cap T = \{3\}$

2 Multiple-Choice: Fine Shades of Meaning

Examine carefully the algebraic expressions below. In each case select the statement from the key which is most appropriate to the expression.

KEY

A true for all values of x

B false for all values of x

C false for all negative values of x

D false for all positive values of x

E false when x lies in the range 0 to 1 inclusive

(1) $x < x + 2$

(2) $x > 2x$

(3) $x^2 > x$

3 Open-Ended Form

Three whole numbers which divide into 20 to give a result which is an integer are 1, 10 and 20. Name three other such whole numbers which divide exactly into 20.

4 Simple

Which of the following is an expanded numeral for 797?

A (7) (9) (7)

B $7 + 9 + 7$

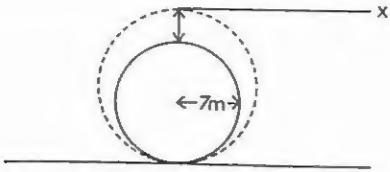
C $7(10)\ (10) + 9(10) + 7(1)$

D $7(10) + 9(10) + 7(10)$

E none of these

5 Complex

A copper ring just fits a large upright wooden wheel of radius 7 metre when the temperature is at freezing point. At boiling point the radius of the wood has not changed significantly, while the length of the copper ring has increased by 4.4 cm.



Which of the following is the best estimate of the distance x, the maximum distance of the ring from the wheel?

A 4.4 cm
B 2.2 cm
C 1.4 cm
D 0.4 cm
E 0.2 cm

6 Naive

$\{1, 2, 3\}$ represents

A an infinite set.
B a finite set.
C a cardinal number.
D the number three.

7 Sophisticated

Consider the following statements:

(i) $ad = bd$
(ii) $na = ma$ for some integers m and n
(iii) $a = b + nd$ for some integer n
(iv) $\dfrac{a-b}{d}$ is an integer

(1) Which one or more of the above statements is/are equivalent to the statement $a \equiv b \pmod{d}$?

(2) For which one or more of the above statements could we be certain that $a \equiv b \pmod{d}$?

8 Simple Recall

In which one or more of the following do the diagonals always bisect each other at right angles?

A parallelogram
B rectangle
C square
D rhombus

9 Advanced Analytical

Exactly 27 per cent of pupils of a certain school are boys. Exactly one third of the pupils have red hair. Even if the number of pupils of the school were doubled, there would be less than a thousand pupils.

(1) What is the least possible number of red-haired girls at the school?

(2) What is the greatest possible number of red-haired girls at the school?

10 Intuitive Ingenuity

Questions 1 and 2 refer to the following diagram and information:

A and B represent identical carts

Cart A rolls along a table at a speed of 12 centimetre per second. Cart B is initially at rest. As soon as the end of spring S (projecting in front of A) touches B, B starts to move. B moves in all a total distance of 6 centimetre from its initial position before the spring no longer touches it. At the instant this occurs, cart A has become stationary, and remains so, and cart B now has a speed of 12 centimetre per second.

(1) The total time during which the spring contacted cart B was 2 seconds. At what time after the initial contact was the spring probably most compressed?

A 0.5 second
B 1 second
C 1.5 second
D 2 second
E at all times during the contact

(2) The carts had the same speed

A when cart B was 3 centimetre from its initial position.
B at some point when cart B was less than 3 centimetre from its initial position.
C at some point when cart B was more than 3 centimetre from its initial position.
D at no stage during the collision.

CHAPTER 2

DIAGNOSTIC TESTS

Use of Tests

As teachers, one of our functions is to help students to learn to master the content and processes that comprise our subject. To know, at the end of a year, through the use of an achievement test, that we have succeeded remarkably or failed lamentably, may be pleasurable or distressing; but so far as active student learning is concerned, it is sterile information. At the year's end, the student's school learning ends also, and it is too late for remedial treatment.

It is clear that, if we are to locate the individual difficulties and misconceptions of students and do something effective about such problems, then continual testing is necessary in order to diagnose errors, and continual remedial work is necessary to remedy these errors.

Diagnostic tests, therefore, are an intrinsic part of the teaching-learning situation. With their help we can isolate our own failures, confirm our own successes, and continually improve our methods in the light of feedback information which the tests give.

But more important than this, diagnostic tests enable students to determine the limitations and discrepancies in their own learning. If the tests are used and followed up in a proper manner, this knowledge is the springboard to future mastery.

Follow-Up

Diagnostic tests are used to diagnose breakdown in student learning. There may be some areas where breakdown is general, where the errors are common to most members of the class, so that the class will benefit from some group reteaching. This may be more effective if a slightly different approach for a topic is used when the topic is retaught.

But often the errors are idiosyncratic ones, and differential materials or approaches will be needed to cope with these very

individual errors and misunderstandings.

The form of follow-up which will be most effective depends on the nature of the problem, the nature of the child, the nature of the teaching agencies (in which I include material and teacher), and the mutual interactions of these three variables. So the problems are complex, and there are no simple answers.

For example, the basic problem may be in the recall of specific facts, in the comprehension of some basic principle, in the method of attack on a coherent set of problems, in lack of care in numerical calculation, and so on. From this brief list it is obvious that a different method would be required to improve each of these deficiencies. Some suggested remedies are illustrated below.

Diagnosis	Possible Remedial Action
Recall of specific facts	Repetition and overlearning and grouping of facts
Comprehension of basic principle	Teacher explanation, reference to another text, programmed instruction
Methodology of working a set of problems	Worked examples followed by a set of practice examples
Lack of care in numerical calculation	Institute methods of checking accuracy

Intensive Diagnostic Tests[1]

How does one obtain information about the difficulties of the individual student from diagnostic tests? Tests are like computers. They give us the information we programme them to give. If we want to know whether children remember the specific basic facts, we must ask questions about those facts. If we want to know whether they comprehend the basic principles, we must ask direct questions about these principles. If we wish to distinguish between careless errors and lack of understanding, we must ask a number of questions about the same basic idea. If a student answers four

[1]These superscripts refer to the sets of questions presented at the end of the chapter.

out of five questions about a particular idea correctly and the fifth incorrectly, the wrong answer was probably due to a careless error. If the student answers one out of five questions about a particular idea correctly, he probably does not understand the idea. If he gets two or three right out of five, both issues may be involved.

Intensive diagnostic tests, then, have much in common with linear programmes.* The elements that comprise the final behavioural task must first be analysed, so that the questions may slowly peck their way through the basic terminology, concepts, and relationships and skills that are involved in that final task. Thus they can indicate the specific points where a student's mastery of the idea breaks down. Only when this is known can adequate treatment be given.

In structuring and writing a diagnostic test, then, we should have in mind the analysis and use of the information we will obtain from it.

Extensive Diagnostic Tests[2]

Extensive tests are used to cover a much broader area of content than are the intensive tests. Whereas intensive tests could be used to test the work of one to four single class periods, the extensive test is used to diagnose the effectiveness of, say, a term's work. Where intensive tests have been used, and followed up by adequate remedial work, the results on such extensive tests should be gratifyingly good. Whilst waiting for Utopia, however, we will find such extensive tests very necessary.

Extensive diagnostic tests are characterized by carefully graded sampling of the work covered by the tests. Good sampling is necessary so that broad areas of difficulty may be indicated, and grading of questions within each area is necessary to hint at the actual point at which a given student's mastery breaks down.

Extensive tests may also test across topic areas, and examine the relationships between broad ideas that have previously been tested in isolation. It is obviously necessary to coalesce diverse mathematical content and processes into some sort of integrated mathematical structure in the mind. Whilst such cross-linkages will occur spontaneously with some students, others may need to be taught definite procedures which will help them to utilize information from different topic areas to solve relevant problems.

*Most programmed texts or courses for use in teaching machines use linear programming techniques.

IDEA OF CLOSURE

PRIOR ANALYSIS

1	Basic terminology	:	Set, element, counting number, fraction, integer.
	Skills	:	Be able to classify numbers or sets of numbers into the above categories. Comprehend the classification of zero in the above categories.
2	Terminology	:	Operation, closure.
	Skills	:	Comprehend the meaning of these terms.
3	Relationship	:	'Set . . . is closed under certain operations.'
	Skills	:	Comprehend the meaning of this statement. Translate between general statements about sets and closure and statements about specific numbers. Infer the truth or falsity of statements about closure of specific sets under specific operations. Apply the idea of closure to new problems.

INTENSIVE DIAGNOSTIC TEST ON IDEA OF CLOSURE[1]

1 How many elements has the set $\{2, 3, 4\}$?

2 How many elements has the set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$?

3 How many elements has the set of positive fractions?

4 Which one or more of the following elements belongs to the set of positive fractions?

A $\frac{1}{2}$
B $\frac{2}{3}$
C 1
D 0
E $-\frac{1}{2}$
F $-\frac{3}{2}$

5 Which one or more of the following elements belongs to the set of negative fractions?

A $-\frac{3}{2}$
B $-\frac{1}{2}$
C -1
D -0
E 0
F $\frac{1}{2}$

6 Which one or more of the following are elements of the set of counting numbers?

A 0
B 1
C 2
D -1
E -2
F 1 000 001
G $\frac{1}{2}$
H $-\frac{3}{4}$

7 Which one or more of the above (in Question 6) are elements of the set of integers?

8 Zero is an element of the set of

A counting numbers.
B positive fractions.
C fractions.
D integers.
E none of the above.

9 One (unity) is an element of the set of

A counting numbers.
B positive fractions.
C fractions.
D integers.
E none of the above.

10 Which one or more of the following symbols represents an operation?

A {............}
B $\div$
C $-$
D $+$
E $\times$

11 Which one or more of the following words describes an operation?

A set
B multiplication
C fraction
D integer
E subtraction
F element

12 Consider the statement: 'The set of counting numbers is closed for the operation of addition.' Another way of saying the same thing is

A if a and b are any two counting numbers, then $a + b = b + a$.
B the sum of any two counting numbers is itself a counting number.
C if a and b are any two counting numbers, and $a + b = c$, then $c - a = b$.
D none of the above.

13 Consider the statement: 'The set of counting numbers is closed under multiplication.' Another way of saying the same thing is

A if a and b are any two counting numbers, then $a \times b = c = b \times a$.

B if a and b are counting numbers, then $a \times b = c$, and c is a counting number.

C if a and b are any two numbers, then $a \times b = c = b \times a$, where c is a counting number.

D if a and b are any two counting numbers, then $a \times b = c$, where c is not a counting number.

Questions 14-16

Consider the statement (referred to below as statement Y): 'The set of integers is closed under division.' This statement may be true or false.

Consider also the four following true statements:

A $8 \div 4 = 2$	**C** $8 \div 8 = 1$
B $8 \div -4 = -2$	**D** $4 \div 8 = \frac{1}{2}$

14 Which one or more of the above (A, B, C, D) support(s) statement Y?

15 Which one or more of the above (A, B, C, D) does/do not support statement Y?

16 From the above information, it may be concluded that

A statement Y is certainly true.

B statement Y is certainly false.

C statement Y may be true or it may be false.

17 Consider the following statement: 'The set M is closed under the operation of division.' This statement is true if M is the set of

A positive whole numbers.

B positive fractions.

C negative whole numbers.

D negative fractions.

E positive and negative fractions.

(one or more answers)

18 Consider the following statement: 'The set N is closed under the operation of multiplication.' This statement is true if N is the set of

A positive whole numbers.

B integers.

C negative fractions.

D positive and negative fractions and zero.

(one or more answers)

EXTENSIVE DIAGNOSTIC TEST ON FRACTIONS, DECIMALS, PERCENTAGES[2]

Simple Operators and Conversions

1 $\frac{4}{5} = \frac{x}{10}$ What is the value of x? . . .

2 $\frac{4}{5} = \frac{2}{y}$ What is the value of y? . . .

Give answers to Questions 3 to 10 in simplest terms.

3 $1\frac{1}{2} \times \frac{3}{4} = \ldots$

4 $2\frac{3}{4} \times 1\frac{1}{3} = \ldots$

5 $1\frac{1}{4} + \frac{3}{8} = \ldots$

6 $3\frac{3}{4} + 1\frac{5}{8} = \ldots$

7 $2\frac{3}{8} - \frac{1}{4} = \ldots$

8 $10\frac{1}{4} - 3\frac{3}{4} = \ldots$

9 $\frac{5}{6} \div \frac{5}{3} = \ldots$

10 $\dfrac{1\frac{1}{2}}{3\frac{1}{4}} = \ldots$

11 $1{\cdot}2 \times 3 = \ldots$

12 $\cdot 3 \times \cdot 02 = \ldots$

13 $\cdot 2 + 1{\cdot}7 = \ldots$

14 $1{\cdot}4 + \cdot 07 = \ldots$

15 $3{\cdot}8 - 2{\cdot}6 = \ldots$

16 $5{\cdot}4 - \cdot 95 = \ldots$

17 $6{\cdot}4 \div 8 = \ldots$

18 $\cdot 36 + \cdot 6 = \ldots$

19 $\dfrac{1}{0{\cdot}05} = \ldots$

20 $2\% = \frac{2}{x}$. What is the value of x? . . .

21 Write 35% as a fraction in its lowest terms. . . .

22 $\frac{1}{5} = n\%$ What is the value of n? . . .

23 $\frac{7}{8} = p\%$ What is the value of p? . . .

24 25% of £8.00 = . . .

25 2% of £6.00 = . . .

26 31% of 64 000 = . . .

27 $0{\cdot}46 = \ldots\%$

28 $2{\cdot}5\% = 0{\cdot}\ldots$

29 $0{\cdot}07 = \frac{7}{x}$ What is the value of x? . . .

30 Write $\frac{1}{8}$ as a decimal fraction . . .

31 What percentage of £12 000 is £6.00?

Analysis and Follow-Up

The structure of this Extensive Diagnostic Test is apparent.

Questions 1 to 10 deal with the four simple operations applied to fractions.

Questions 11 to 19 deal with the four simple operations applied to decimals.

Questions 20 to 31 deal with conversions among decimals, fractions and percentages.

A set of 10 similar questions, graded for difficulty, could be produced for each question in the diagnostic test, perhaps with one or two worked examples. Any student with a wrong answer would be required to do the appropriate set of 10 questions, asking for help as he felt necessary.

If errors were still made, analysis of these more intensive tests should reveal the nature of the difficulty.

Set of Graded Questions for Question 4

Worked Example: $2\frac{3}{4} \times 1\frac{1}{3}$

$2 = \frac{2}{1} = \frac{2}{1} \times \frac{4}{4} = \frac{8}{4}$ (multiply fraction by $\frac{4}{4} = 1$)

So $2\frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}$

$1\frac{1}{3} = \frac{3}{3} + \frac{1}{3} = \frac{4}{3}$

$$2\frac{3}{4} \times 1\frac{1}{3} = \frac{11}{\cancel{4}_{1}} \times \frac{\cancel{4}^{1}}{3} = \frac{11}{3} \qquad (\tfrac{4}{4} = 1)$$

$$= 3\frac{2}{3}$$

1 $\frac{1}{3} \times \frac{2}{3}$
2 $1\frac{1}{3} \times \frac{1}{5}$
3 $1\frac{2}{3} \times \frac{3}{8}$
4 $2\frac{2}{3} \times 1\frac{1}{4}$
5 $1\frac{1}{3} \times \frac{3}{4}$
6 $3\frac{3}{4} \times 1\frac{1}{5}$
7 $\frac{2}{3}$ of $3\frac{1}{2}$
8 $\frac{1}{4}$ of $\frac{1}{4}$
9 $2\frac{1}{2} \times \frac{3}{4} \times 2\frac{2}{3}$
10 $\frac{3}{4} \times 1 \times 1\frac{1}{5} \times 1\frac{1}{3} \times \frac{5}{6}$

CHAPTER 3

ACHIEVEMENT TESTS

Achievement Tests and Standards

Most people who set examinations, from Grade 1 to university, believe they are testing achievement. They have in mind some *standard* of achievement and believe (often with an innocent and touching faith) that their test will assess whether this standard has been reached. A great number also believe that this acceptable standard is located as a mark on the test. This mark is often defined as exactly one half of the total number of marks available on the test. Nowhere (except in the universal abhorrence of Friday 13) is faith in the magic of numbers so deep-seated as on this matter. Nowhere is a dogma that is so easily refuted so vociferously upheld. Whenever such 'absolute standards' of achievement have been analysed closely, it has been found that the scale, which the examiner believes is marked on an invariant linear axis, is in reality etched on an unpredictable topological surface* (see Figure 1).

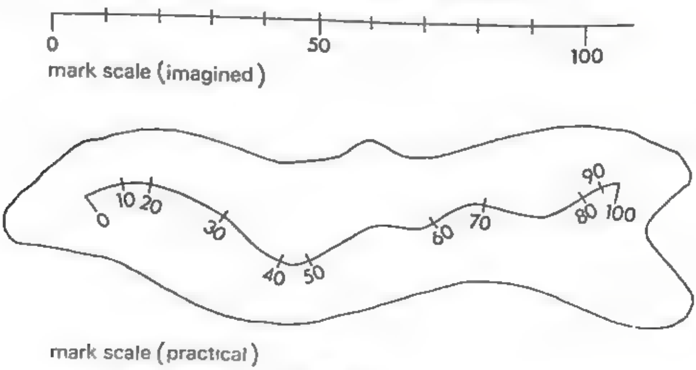


Figure 1

*The classical study in this area was done by Hartog and Rhodes in 1935—*The Marks of Examiners.* It was shattering, so shattering that most investigators have shied away from the topic since.

But even if we could assume that equal mark intervals did represent some invariant intervals on the achievement dimension, the mystique that the fraction $\frac{1}{2}$ has some magical significance in regard to achievement is intractable to logical analyses. The mind, as Wodehouse would say, boggles. The practice of using one half of the total marks as a 'pass' mark can be understood, I suppose, in terms of its mathematical elegance or its administrative simplicity. But it must be abhorred in terms of its educational incompetence and its statistical unreality. My own positive and serious recommendation is that the words 'pass' and 'fail', being so often logically and educationally and statistically indefensible, should be banned from educational discourse, and the concepts they represent be completely rethought.

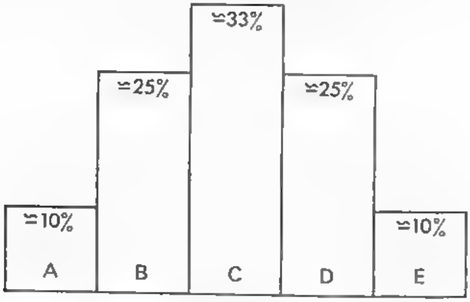
Achievement Tests and Ordering

If we are to view with some suspicion the actual mark obtained in an achievement test, can we place more reliance on the ordering of students that it produces? The answer is yes. There is evidence that the rank orders produced by achievement tests are more likely to be invariant than actual percentage scores gained—though perhaps, again, far less reliable than most people are willing to believe.

Where large numbers of students are involved (say 200 or more), and where there is no reason to believe that either student abilities or learning conditions have changed appreciably from one year to the next, it is reasonable to assume that the distribution of final attainment of successive groups is relatively constant from year to year. With this assumption, it is reasonable to give students a grade (say from A to E) on the basis of their rank order on the achievement tests.

It should be noted that the grades must be *defined* by the percentage of students included in each grade, and not by the subjective opinions of examiners. Only by some such definition in terms of percentage distribution of students may the grade take on some stable meaning.

One such distribution is shown in Figure 2. In this case a grade of A is defined as the attainment of the top 10 per cent of a certain student population, a grade of B is defined as the next 25 per cent of students, and so on. It should be noted that these grades do not refer to any absolute standard of achievement.



Sample distribution for defining grades
Figure 2

How to Misinterpret Statistics

What has just been said about grades refers only to large groups with an unchanging composition. The argument may *not* be applied to small groups, such as a single class or even two or three classes. It is dangerous to apply the method even to a larger group where the group is non-representative, for example, to a school drawing most of its members from a particular socio-economic group.

We could of course define the grades in terms of the population of a particular school, but then we could not compare grades from school to school. Unless this is possible the meaning of the grades, if not obscure, is at least very limited.

So it seems that the grading system, which at first appeared to hold the solution to our problem of stable assessment of students, on closer inspection offers little assistance to the classroom teacher who wishes to assess his particular group of students.

With a privately prepared test of unknown statistical characteristics, and with a sample of students which is not a random sample of all students, the teacher is certainly not justified in using a grading system which refers (at least implicitly) to the results of random groups on supposedly parallel tests.

Mastery and Achievement

But there is a much more cogent argument against the use of achievement tests of the type currently used. Pass-fail dichotomies, and grading systems in a less obvious way, build educational failure into the educational system. Traditions are built up regarding expectations of failure rates in courses. If the pass rate rises, this is often seen as a lowering of standards, rather than an improvement in

teaching or a fortuitous improvement in the quality of student. (How would you feel about a colleague who, teaching an ungraded class, passed all of his students?) In practice, if grades are used, 30 or 40 per cent of students inevitably receive a D or E rating, which they and others will, because of the pass-fail mentality of our system, equate with failure.

Carroll and Bloom, amongst others, have proposed that, given sufficient time, all students can achieve mastery of a mathematical topic (or a topic in any other subject, for that matter). Some difficult topics may require so much time for some students to master that it would be quite unrealistic for those particular students to attempt that particular topic. Nevertheless there is some evidence that very good performances can be produced on many topics by students who are generally considered mediocre or poor, provided such students are given appropriate educational treatment and sufficient time.

The description 'slow', then, is not a euphemism for 'stupid', but an accurate assessment of the fundamental characteristic which, in our educational framework, leads to failure.

When educational programmes are based on personal rather than group goals, this inevitability of failure disappears. 'Slow' students become successful students; but they achieve success in a more limited range of topic areas, because the time they must spend on each topic in order to achieve mastery is that much longer.

Components of Achievement Tests

Analyses of achievement tests suggest that they are usually composed of two rather distinct classes of questions. The first type of item would be relevant to tests of mastery, to be examined in detail in the next chapter. The second type of item, which involves general mathematical problem solving, is discussed in Chapter 5 under the heading of 'Tests of Mathematical Learning'.

For reasons already hinted at, I believe it is educationally desirable not to compound these two classes of item in a single test paper. Much of the abortive discussion on the meaning of examination marks and much of their educationally disastrous effects are due to this fundamental confusion.

CHAPTER 4

MASTERY TESTS

What Is a Mastery Test?

In this book we shall define a mastery test as one which is based on the tightly prescribed mathematical content of a course and does not attempt to measure students' ability to process unknown mathematical material or to apply the principles learned to problems of an unknown type.

What sort of questions might be asked in such a mastery test? The following are some descriptions of relevant aspects:

(a) questions testing recall of basic mathematical facts or generalizations or procedures.[1]

(b) questions testing comprehension of the mathematical structure of the course topics.[2]

(c) questions testing ability to do examples similar to those worked during the course, questions which exemplify the mathematical performance the course aims at.

The importance and loading given to each of these aspects on the test would be dependent on the stated aims of the course. Where syllabus writers have defined the limits of the course unambiguously, the relative importance of each of these should be explicit. In practice, test writers generally have to produce their own loadings both for topics and skills. Procedures for doing this will be discussed in Chapter 9.

In appearance, a mastery test may be very similar to an extensive diagnostic test. The interpretation and use of its results are, however, very different, and the questions are more concerned with terminal behaviours than with learning difficulties.

When Is There Mastery?

Students, being human, will always make 'careless' errors in

[1]These superscripts refer to the sets of questions presented at the end of the chapter.

thinking and in calculation. So mastery does not need to imply perfection. Nevertheless it may be possible for all students to reach a level where they can successfully complete 80 or 90 per cent of the tasks that are considered to comprise an adequate sample of expected performance in a given topic. Many good primary teachers achieve this sort of performance from their children by manipulating time spent on mathematics to suit the needs of particular pupils. Such manipulation of time is easier to achieve in the looser class period structure of the primary school.

Topic Mastery Tests

It does seem feasible then to produce topic mastery tests which students could attempt when the teacher in his wisdom, the child in his enthusiasm, or the remedial test in its statistics, indicates that the time is ripe. Students would proceed to a new topic only when mastery of a given topic was achieved. These topic mastery tests could be produced by panels of experts at a central testing agency or through the efforts of local groups as described in Chapter 9. They would represent goals which were attainable by all students at a high level of performance, given sufficient time and sufficient diversity of teaching methods and materials. They would both eliminate the necessity for failure at present inherent in our system, and at the same time provide motivation and satisfaction for students.

I do not think such mastery tests could, or should, include questions testing ability to apply the knowledge to new situations which differ significantly from previously worked examples. It is my personal belief that such tests of 'problem solving' or 'mathematical learning ability', in as much as speed and facility are major components, will always produce wide distributions of scores. But if the mastery tests are limited to questions based on ability to comprehend specifically taught material, or to successfully work examples similar to those studied in detail during the course, they should be produced, and the educational structure should be modified so that all children can achieve success in some mathematical topics.

Let's Be Realistic

If you have followed the argument this far, I can almost hear your cries of anguish. 'I've got a syllabus to cover, you are talking idealistic rubbish.'

I do not wish to minimize the difficulties in the approach I am suggesting, and in Chapters 9 and 10 the practical solutions will be discussed in more detail.

At this stage I would ask only this: even if you have been presented with a syllabus, is it better to attempt to push all students through it, knowing that there will be many failures with the concomitant emotional rejection of the subject? Or is it better for these slower students to cover only part of the syllabus, and achieve mastery and satisfaction from those parts successfully completed?

Is realism surrender to a system which is inadequate, or is it the taking of effective action to modify or remedy defects?

QUESTIONS TESTING RECALL OF BASIC MATHEMATICAL FACTS OR GENERALIZATIONS OR PROCEDURES[1]

1 In which one or more of the following do the diagonals always bisect each other at right angles?

A parallelogram
B rectangle
C square
D rhombus
E kite

2 Which one or more of the following is *not* a member of the set of quadrilaterals? If all are members of the set, write E.

A square
B rectangle
C rhombus
D kite

QUESTIONS TESTING COMPREHENSION OF THE MATHEMATICAL STRUCTURE OF THE COURSE TOPIC[2]

Topic: Simple arithmetic calculations. Five fundamental laws of arithmetic are stated below in symbolic form:

A $a + b = b + a$
B $ab = ba$
C $a + (b + c) = (a + b) + c$
D $a(bc) = ab(c)$
E $a(b + c) = ab + ac$

1 Consider the statement

$$8 \times 9 = 9 \times 8$$

Of which law (A, B, C, D or E) is this a special case?

2 Consider the statement

$$6(12 + 5) = (6 \times 12) + (6 \times 5)$$

Of which law (A, B, C, D or E) is this a special case?

3 We cannot state a general law in the form $4 + 6 = 6 + 4$ because such a statement

A would not include fractions.
B is obviously true for all numbers.
C may be extended to provide many other rules.
D does not imply that the same rule holds for other numbers.
E is meaningless.

4 Which of the following could *not* be justified by one of the five laws?

A $m \times (p \times q) = (m \times p) \times q$
B $m \times (p + q) = mp + mq$
C $m + (p \times q) = (m + p) \times (m + q)$
D $m + (p + q) = (m + p) + q$

5 The laws of arithmetic may be stated using the symbols a, b, and c. Which one or more of the following may these symbols represent?

A positive integers
B negative integers
C positive fractions
D negative fractions
E zero

CHAPTER 5

TESTS OF MATHEMATICAL LEARNING

What Is a Test of Mathematical Learning?

By our definition of mastery tests we limited the content of such tests to items based on specific course content: questions similar to those studied during the course, questions testing comprehension of the mathematics of the topics taught.

In a test of mathematical learning, the emphasis is on facility of learning, the very factor we wished to minimize in the mastery test by giving students varying teaching treatments, and varying intervals of time in which to master the topic on which the mastery test was based. For example, such tests of mathematical learning may include items based on completely new mathematical material; or a new mathematical operator or system may be explained and the student be then required to answer a number of questions based on this new operator or system; or problems may be asked requiring the student to apply concepts previously learned to completely new situations.

To answer such questions successfully, some 'instant learning' is necessary. In this sense mathematical problem solving is another aspect of quick and imaginative mathematical learning.

Is It Possible to Teach Problem Solving?

Now it may well be possible to teach directly and effectively the heuristics of problem solving so that all students can achieve mastery in this area. Certainly assiduous concentration on styles and techniques of solution and proof in special mathematics topics may improve a student's general problem-solving ability. And certainly we, as mathematics teachers, might well spend more time attempting to teach, directly, such heuristics.

But very little is known about how to go about this, and for some time to come I would expect a student group to exhibit a wide range of scores on such a test.

Polya has attempted to isolate some of the analytic heuristics used in solving mathematical problems, but very little research has been carried out at a more elementary level. So for the present we must accept the fact that few children will perform well in problem-solving situations as we have defined them here.

If we examine the nature of the task, this is surely to be expected. For problems in mathematics are of an infinite variety—in complexity, format, style, profundity, imagination and elegance. There is no perfect mathematician, just as there is no perfect athlete. So it is always possible to separate the performance of any two students by taking an adequate sample of mathematical tasks at an appropriate level of difficulty.

Risking an analogy, we could use football skills to mirror the difference between mastery tests and tests of mathematical learning. Given sufficient time and sufficiently expert and varied direction, any normal child can be taught to kick a football a distance of 30 metres with reasonable accuracy. In a like manner any child can be taught to perform adequately the specific tasks required in a mastery test. But few children, regardless of their training, have all the qualities necessary to make an international football star. In integrating all the many specific skills, in reacting swiftly to the ever-changing complex situation, the 'natural' potentialities of the individual assume more importance, even though specific training is very necessary. Similarly there are few among us with the potentiality for mathematical genius—though there may be more than we imagine.

Tentative Classification of Skills

Test constructors are fond of talking about skills. These skills are imagined to be the cognitive components of mathematical performance. They are extruded in a valiant attempt to create some order out of the complexities of the situation. As tentative crutches to test writing and curriculum development they are useful, but we must beware of ascribing to them more permanence and reality than they deserve.

Five descriptions related to problem-solving 'skills' are given below. They are not mutually exclusive nor are they necessarily all-inclusive. They are merely useful.

(a) Comprehension of new mathematical material—the new material may be a definition, or a description of a new operator or system.

(b) Problem solving in structured situations—the mathematical situation is not completely new, and the problem is so structured that the method of solution is hinted at.

(c) Problem solving in unstructured situations—the problem is so stated that little or no hint is given regarding a possible mode of solution.

(d) Ability to follow and construct proofs—normally only analytic aspects of proof may be tested in the normal objective tests, which do not lend themselves to testing ability to synthesize a proof.

(e) Invention of tentative intuitive solutions—it is very difficult to set objective questions in this area, but some attempts have been made.

The intrinsic difficulty of a problem also depends on the form in which it is presented, and the form in which the answer is required.

Problems may be set in forms which are predominantly

(a) verbal
(b) symbolic
(c) spatial
(d) graphical
(e) numerical.

The form of solution may likewise be in any of these forms. The complexity of transformation among these various modes of presentation is thus a very important aspect of the problem-solving situation.

In Appendix II of this book is a collection of items which could be used in tests of mathematical learning. Some of the easier items would be suitable for Form 2 students, all are suitable for Form 4 students, and most would discriminate quite well among Form 6 students. Their relative difficulties for Form 4 students (based on results on the Commonwealth Secondary Scholarships Examinations from which they have been taken) are indicated on a 1 (easy) to 5 (hard) scale.

A detailed list of test categories follows as a useful aid to test item writers.

DETAILED LIST OF TEST CATEGORIES

Categories of Knowledge (Content)

Terminology (e.g. standard symbols, definitions)
Specific Facts (e.g. number facts)
Conventions (e.g. order of operations)
Trends and Sequences (e.g. graphs and scores)
Criteria (e.g. assumptions)
Methodology (e.g. processes involving equalities and inequalities)
Principles and Generalizations (e.g. theorems, formulas)
Theories and Structures (e.g. laws of arithmetic)

Categories of Skills

Comprehension

Translate from one form to another
Verbalize symbolic and geometrical relationships
Illustrate mathematical concepts, terms, principles, and processes by giving examples
Explain mathematical concepts, terms, principles and processes in own words
Detect errors in definitions, processes, proofs
Identify mathematical relationships in familiar problem situations
Compare related concepts, processes, figures
Discriminate among closely related terms, processes
Verify results
Choose an appropriate technique
Manipulate symbols to an appropriate algorithm or technique
Estimate a numerical answer

Analysis and Application

Establish relationships among the data
Reduce an unfamiliar situation to a familiar one (seeing patterns, isomorphisms, symmetries)
Judge the adequacy, inadequacy or superfluity of data
Judge the correctness of a proof (validate the solution)
Select the most appropriate formulas, methods or processes to solve a problem
Solve a problem by using the appropriate facts or processes
Generalize from data
Infer from data

Synthesizing

Formulate problems
Construct proofs
Present a solution in a mathematically elegant manner

CHAPTER 6

PREDICTIVE TESTS

What Are Predictive Tests?

Mastery tests are designed to demonstrate student mastery of a given topic or a group of topics. Tests of mathematical learning are designed to measure the facility of students in complex problem-solving tasks; predictive tests are designed to predict future performance or to demonstrate suitability for further training. To prepare adequate predictive tests we need to know details of both the nature of the performance to be predicted, and the training methods that will be used to produce that performance.

Content Validity of Predictive Tests

Often we are concerned to know whether the student has adequate grasp of that knowledge which is a *prerequisite* for the next phase of the course.

An analysis of the real function of an end-of-the-year examination reveals that, administratively, this is perhaps its chief function. The administrator wishes to know, 'Has this child sufficient knowledge and skill to cope with next year's syllabus?'

To answer this question we should not be looking back at the past, but forward to the future. We should be analysing the implied prerequisites of the next year's topics, and set our test accordingly. Perhaps this type of test is more accurately termed a 'readiness test'. It may be, of course, that this information is already available in terms of mastery tests done throughout the year. If so, the readiness test becomes superfluous.

'Learning' Validity of Predictive Tests

We also need to know how the next phase of the course is to be taught. Are individual students to be allowed to progress on each topic at their own optimum rate, with diagnostic and remedial help, finally to demonstrate mastery of the topic? Or are they to be

treated as a group, so that facility in learning becomes more important?

To the extent that facility in learning mathematics is an important component of the performance to be predicted, to that extent will tests of mathematical learning be important in predicting that performance. If undifferentiated group instruction is given for a crowded course of study, then results on tests of mathematical learning may correlate highly with final performance.

But if, during the next phase of learning, students are to be given differential treatment so that all may hope to attain a preconceived mastery of clearly defined tasks, then tests of mathematical learning will not be efficient predictors of success. Differing treatments result in a levelling out, in a uniform mastery, whereas undifferentiated treatment places a premium on facility.

It could be concluded that, where mastery is the major aim of a course of instruction, then the higher the correlation between a predictive test and the results on a course, the more inefficient the teaching of that course is proved to be. For those who regard prediction as the major criterion for a terminal test, this should be an interesting speculation.

CHAPTER 7

GENERALIZABILITY

A test is a test of something. If I am asked what it is that a test I have just produced is supposed to be measuring, I might like to say, 'Here's the test. Read the questions and decide for yourself.' And there are some test papers, particularly experimental ones dealing with complex learning tasks, where the most honest appreciation of the objectives of the paper is best gained in this way. But in general it is the coward's way, a pusillanimous slide from responsibility.

Usually we know perfectly well what it is a test of. It is a test of students' comprehension of a specified topic or syllabus. Or it is a test of students' general mathematical ability. Or it is a test of students' ability to proceed to further study. Or, often, it is a test which pretends to measure all of these concurrently, and probably succeeds in measuring none very well.

The fact that a test is a *measure* involves us in ideas of a scale and in ideas of accuracy. The fact that a test is a measure of *something* with many and varied aspects involves us in ideas of sampling from a conceptually coherent universe of questions. The fact that a test is a *measure of something* involves us in ideas of the *stability and meaning* of our result.

All of these ideas are involved in the idea of generalizability. Broadly, the idea is this: given the objectives of a particular topic or course, or the teaching or learning prescriptions, one can imagine a very large number of questions that could be asked, all more or less relevant, all more or less representative of the sort of performance we can expect from the student. For a given test, we must sample from this large potential universe of questions so that the student performance in miniature as he performs on the test accurately mirrors his performance were he to attempt all such possible questions, and were we to load each aspect of his total

performance to conform to our preconceived idea of their relative importance.

Furthermore, if one such test is produced by one person or group, and another such test is produced independently by another person or group to the same *stated* course prescription, the results obtained by students on the two tests should be equivalent or at least close enough to include an acceptable margin of error. This is what we mean by generalizability: the confidence with which we can generalize from one set of test results to other sets of test results based on independent tests produced to the same basic prescriptions. And the confidence with which we can so generalize to other tests is the confidence with which we can describe the test as one of comprehension of the topic, or mastery of a skill, or whatever it purports to be.

Sampling

It is apparent that the time available for testing is a limiting factor in effective sampling. In Chapter 1, we mentioned that one of the advantages of the objective format for tests was that it enabled us to ask many more questions in the same time than in an essay type of examination, and so allowed us to sample the total universe of questions more adequately. (Proponents of essay-type questions might like to argue that students who 'play the game' properly sample the content and skill field in their answers. This is a valid point. It may sometimes be true.)

If possible, the relative importance of various topics (or aspects of topics), and of the various sorts of mental processing involved, should be decided before the paper starts to evolve. In mastery tests, where the end performances are relatively clear-cut and specific, this is not too formidable a task. But even here it is necessary. It is all too easy to compose a test of questions which are easy to set rather than of questions which are important. It is so much easier to test recall and to set standard problems, and neglect to test comprehension of the fundamentals that underlie the method.

It is more difficult to produce an acceptable preconceived plan for a test of mathematical learning. But it is even more necessary, for the same reasons we have mentioned above.

Having stressed the necessity for preplanning, however, we need to go one step further. The plan itself is based on tentative value judgments, and is far from omnipotent. Furthermore a test is more than the sum of its parts. The good test has a coherence of structure.

At the risk of appearing extravagant, I would claim that the really excellent test is, in its way, a work of art. After the test is produced, it is not the analytic plan that should decide its ultimate success, the proof of its adequate sampling. Rather it is the looser and yet more relevant criterion, the answer to the question, 'Does this paper, taken as a whole, accurately reflect the aims and objectives of the learning situation?' If the answer is honestly yes, then the test may be a good test. If the answer is no, then regardless of any congruence with a sampling plan, the test is inadequate.

A Measure of Generalizability

This discussion of the nature of generalizability suggests how it should be measured. If two hours are available for testing, two test papers, each of one hour duration, should be produced independently on the basis of the most specific prescriptions available.

These tests should preferably be administered at different times. For some groups give one paper first, for other groups give the other paper first, if this is possible from the point of view of administration and security. The correlation between the two sets of results is then a rough measure of the generalizability of the test.

CHAPTER 8

ANALYSIS OF A TEST

Item Difficulty

It is almost impossible to predict how difficult a test question will be before it is tried out on a group of students. Particularly is this true of the sorts of items written for tests of mathematical learning. So after testing, each item should be checked to find what percentage of students obtained the correct answer. Our expectations will vary with the type of test.

In a diagnostic test we might expect the items to become more difficult as students explore each idea in more and more depth, as they proceed to the final performance task required.

In a mastery test we might expect 80 or 90 per cent of students to get each item correct. Items which prove to be more difficult than this could indicate areas in which there has been insufficient learning or insufficient teaching. Or this could indicate some deficiency (irrelevance or ambiguity) in the item itself.

One of the main functions of a test of mathematical learning is to separate students on their general facility in mathematical problem solving. Now it can be shown statistically that, if we wish to achieve maximum spread of marks from a given number of items, then each item should elicit about 50 per cent of correct responses. It is general practice in such tests to accept items between the 30 per cent and 70 per cent levels. If an item is outside this range we would need very strong extra-statistical reasons for including the item in preference to one with more statistically acceptable characteristics.

In a predictive (readiness) test based on prerequisites for the next phase of a syllabus, we might expect most students to exhibit mastery of the items. If they could not do so, then it would be apparent that the previous course did not teach those tasks to mastery level, and the teachers of the next phase would not be justified in assuming that these prerequisite levels were reached.

Item Discrimination

In tests which claim to measure some generalized ability, it would be expected that each item would discriminate between the more and less able. That is, if on the basis of scores on the whole test we divide the students into an upper group and a lower group of equal size, then for each item more of the upper group than the lower group should give a correct answer.

The simplest statistic to use here is the discrimination index (D), defined as $D = \dfrac{U - L}{R}$ where

U is the number in the upper group who gave correct responses,
L is the number in the lower group who gave correct responses,
and R is the number in each group.

If the top 25 per cent are placed in the upper group and the bottom 25 per cent in the lower group, then for items of suitable difficulty (30 per cent to 70 per cent) the values of D are directly comparable in terms of discriminating power.*

For such a test of generalized ability, an item with a higher value of D might be preferred to an item with less discrimination if both items seem equally relevant and important for a particular sort of mathematical skill. Such an item will tend to produce a broader distribution of final scores.

The value of the item discrimination index for diagnostic, mastery and readiness tests is questionable. Because they deal with very specific learned tasks and because the ultimate performance may be reached through different treatment methods and routes, there would seem to be no a priori reason to expect items to discriminate, because there is no reason to expect discrimination.

Pretesting Items

In the production of tests for commercial use it is the practice to prepare many more items than are ultimately required, and then select the final test questions on the basis of the item statistics described above, together with considerations of relevance and balance of the test.

Teachers may find it impossible to follow this practice, desirable as it may be. However, some methods of achieving this sort of pre-knowledge are discussed in Chapter 9.

*Under these conditions the values of D are approximately equal to the bivariate normal correlation coefficient with less than 10 per cent error.

Errors in Estimate

It was suggested in Chapter 7 that, where a function of a test is to assess comparative merit of students on some sort of performance, it is necessary to know to what extent the results on one sample of performance (test) can be generalized to another sample of performance (a second test), independently generated.

This could be estimated by finding the correlation between the two sets of scores if it is distributions we are interested in, or the percentage of students whose position is changed to the other side of the cut-off line if it is some absolute idea of mastery that interests us.

But a correlation coefficient is a rather complex statistic. Probably teachers would find it more useful to know the probable accuracy of the estimated mark.

For example, a score of 60 ± 5 has a different meaning to a score of 60 ± 15 in terms of the confidence we may attribute to the mark. In mastery tests, we have argued that we are vitally interested in the percentage of acceptable performances of the student, so some estimate of error of this sort is called for.

Where we have two estimates of a student's score from two independently prepared tests, the best resulting estimate is the mean of the two marks. And if we wish to define the limits as those within which 95 per cent of such estimated marks are likely to fall, the technique described in the appendage to this chapter gives the sort of rough estimate we are seeking. Whilst lacking statistical sophistication, this method has an honest simplicity that commends itself.

I believe that, if our professional integrity is to rival that of engineers, all measurements in education should include such estimates of error. We owe this sort of frankness to ourselves, to parents, and to our students.

ESTIMATE OF GENERALIZABILITY OF ERROR IN TESTS

Student	Test 1 Scores	Test 2 Scores	Change in Score	Best Estimate of Score
1	23	42	+19	33
2	35	32	−3	34
3	41	49	+8	45
4	46	50	+4	48
5	47	59	+12	53
6	50	42	−8	46
7	51	63	+12	57
8	53	49	−4	51
9	55	73	+18	64
10	59	58	−1	59
11	59	50	−9	55
12	60	71	+11	66
13	61	65	+4	63
14	63	49	−14	56
15	65	63	−2	64
16	67	80	+13	74
17	69	74	+5	72
18	70	65	−5	68
19	71	73	+2	72
20	71	68	−3	70
21	71	78	+7	75
22	74	63	−11	69
23	75	84	+9	80
24	77	71	−6	74
25	78	79	+1	79
26	79	85	+6	82
27	81	89	+8	85
28	85	92	+7	89
29	87	83	−4	85
30	92	90	−2	91

We wish to know the error range within which 95 per cent of marks will fall. As there are 30 students, we may neglect the largest change in score (+19), and accept the next highest (+18) as our 95 per cent value.

As our best estimate of mark is the mean of the first and second marking, however, the expected error from this mean mark is $\frac{18}{2} = 9$. The mark of student 7, then, is given as 57 ± 9.

CHAPTER 9

HOW TO PRODUCE AN OBJECTIVE TEST

At various points in the preceding chapters, Chapter 9 has been held up both as a golden carrot and as a palliative to ease the tensions that arise when the enormity of the task of producing a good objective test becomes apparent. 'This will be discussed in Chapter 9,' says the text, and so the problem is postponed.

Accordingly, whilst you perhaps approach this chapter with some expectation, I start writing it with some trepidation. But one can only try. And the problem is clear, if not the solution. How does a classroom teacher, with limited time and limited experience, produce a good objective test in mathematics of the required type at the appropriate level on the appropriate topic?

Need for a Panel

The first point I would like to stress, and stress again, is this: a teacher working alone, no matter how brilliant or experienced, is most unlikely to produce a really good test. This is an absolute statement. But it is based on some years of professional work in test construction. We know, from bitter experience, that individuals just do not see the ambiguities and faults in their own work. We know that a test produced by a single person has neither the tightness and clarity within items, nor the variety and contrast between items, of a test produced by a co-operating panel.

Where will the panel come from? In general, six or seven is the optimum number for a test panel. If you are in a large school, there may be sufficient numbers in the mathematics department. Even if not all members are concerned with the same level, it may be of some value to have all members working together on tests, so that the coherence (or lack of it) in the whole course will gradually become apparent to all. If you are in a smaller school, there will often be other schools in the area from which a panel may be brought together. When the school is very isolated, some of the early work of criticism can be carried on by letter.

Function of the Test

The first requirement in preparing a test is to decide its function. What is it supposed to measure? Why is it important that this be measured?

The last question is not a facetious one. The production of a test is, as we shall see, a serious business. It is demanding in time and energy and thought, both analytic and creative. So if the test does not serve some very important function in student learning, it should not be produced.

Type of Test

The function of a test will determine its type—diagnostic, mastery, mathematical learning or predictive. If a panel is already constituted, it might be valuable even at this stage to discuss priorities here. Which of these types of test is most needed?

Prescription for Test

As was described in Chapter 7, a test can be analysed in terms of its content, its form, and its 'skills'.

In mastery tests, these aspects are quite explicit if the objectives of the course or topic are clearly laid down. Often, however, such prescriptions will not be available. So one of the first tasks of the panel may be to try to decide just what content is to be mastered, and what is peripheral.

Such discussions of current syllabuses are likely to disclose that there has been no demarcation between what is to be learnt to mastery, and what are generalized skills that are more relevant to a test of mathematical learning. In this case discussion may need to be directed towards this issue—what is the relative importance of mastery of specific topics on the one hand, and the development of generalized mathematical problem-solving skills on the other? Obviously both are important in any course of mathematics. But the balance between them is dependent on both the abilities of students and the basic philosophy of the teacher.

Whilst such discussion of fundamental aims is of great value, no panel need bog down on this issue. If it is decided that both mastery tests and tests of mathematical learning are necessary for a given purpose, then it is not necessary to assume that individual teachers using the test must apply equal loadings to these two aspects, either in their teaching or in the production of some total mark that may

be insisted on by the school administration. Nor, for that matter, is it necessary that a particular teacher use the same loading for different classes.

Some teachers believe that loadings (the percentage of total marks) on a group of questions should be related to the time taken to work such questions in the test situation. This is quite unjustified. Loadings should be a function of relative importance, which is quite unrelated to time taken to test. It may be possible to test one aspect of a topic in two minutes, yet take thirty minutes to test another aspect of that topic. If they are of equivalent relevance and importance, however, their mark loadings should be the same.

Whilst such initial discussions of analytic prescription are necessary prior to the writing of items, there is no need for complete agreement at this early stage. Often the issues will become clearer as the writing and criticism of items progresses.

Writing Stage

The writing of test items is a creative act, so items are produced initially by individual panel members.

Ideas for items may come from studying a rich variety of examples (such as are included in Appendix II to this book), from consideration of difficulties of students, from analytic probings into the development of a topic, from clear specifications about the performance to be mastered, or from intuitive and imaginative sources that may not be specified. The ideas may come in the hour you put aside for writing items, or in a crowded train, or in the middle of a meal, or in the rich environment of sleep.

In books on objective tests, many rules are given about how to write good test items and about the faults that bad items have. My advice is just to write them. Then tear them up if on reflection you think they are not relevant or are puerile. Tidy them up a little if you can. Then submit them to the panel to iron out the ambiguities and faults that they probably still contain.

Discussion Stage

Panel discussions of test items can be shattering to the ego, wearing to the emotions, and confusing to the mind. But, at least in my experience, they are stimulating, extending, and enjoyable.

In the early stages, panel members may feel reticent about criticizing the efforts of their colleagues, who are quick to defend their efforts. This is as it should be. But out of such dissension and argu-

ment comes clarity and, in terms of test items, quality. And gradually such criticism becomes accepted not as a personal attack but as a co-operative attempt to improve, to modify, and to restructure.

Criticism of ideas may be directed at their relevance or their mathematical accuracy, at their ambiguity or their lack of clarity of expression, at their lack of elegance or their over-sophistication and complexity. If the criticism is frank and honest, it will lead to an improved product.

Rewriting Stage

As a result of panel discussion, it will usually be necessary to rewrite the item. Sometimes other items may have been suggested which supplement the ones initially produced. Often two or three rewrites might be necessary before a final set of acceptable items is produced.

Final Assembly of the Test

When sufficient items have been produced, the first test can be put together. If possible, enough items should be available for two tests, so that these can be tried out and the best items selected for a final form. This is probably not a feasible procedure in the normal classroom situation. So another possibility is to make the test about one half as long again as is ultimately required, so a one-hour test becomes a one-and-a-half-hour test on the first occasion it is used. Analysis of the items may then indicate which are not satisfactory, and so the final one-hour paper can be produced and used on future occasions.

The implication here is that tests should, where possible, be kept confidential, so that they may be re-used. The insight given into the effort required to produce a good test makes the reasons for this obvious.

The final paper should be based on the analytic prescriptions initially laid down, and developed by the panel in the light of their discussions. But care should also be taken that the form in which the final test is produced has balance and style, and is interesting. A dull test paper is a bad one.

And the ultimate criterion of the final product is a positive answer to the question—does this test accurately mirror the performance we require of our students?

CHAPTER 10

INTERACTION

Whilst this book is specifically concerned with objective tests, the discussion is relevant to testing, regardless of the particular form of test item.

Testing has been presented here as an integral part of the teaching-learning situation, as an aid rather than an ogre.

The methods suggested for the production of tests have been used for many years at ACER and in similar organizations throughout the world. It is my belief that, until teachers use similar techniques, their tests will not really be of adequate quality.

But the effects of the sort of co-operative panel discussions advocated here go far beyond the mere production of tests. Their reaction upon the teaching situation is far more deep-seated than the mere production of the few sheets of paper comprising the test.

There is inevitably a clarification of teaching aims and objectives. In the frank discussion of item criticism it often becomes apparent that our teaching efforts need modification or redirection. It often becomes apparent that our aims have become stereotyped, and have lost the fluidity and imagination and relevance to student abilities and interests that is so necessary.

Similarly it may be that our teaching methods need more modification and variation than we have been willing to consider.

Another bonus to be gained is the opportunity for personal growth in our own mathematical development that such discussions bring. Through criticism and discussion many mathematical topics reach a new clarity.

And finally we are brought into contact, in an atmosphere of frank co-operation, with people with similar intellectual interests and problems to our own. Out of such a rich texture of intellectual involvement can come both a relief from the professional isolation that accompanies much classroom work, and a positive intellectual stimulation from the company of colleagues.

APPENDIX I

PANEL DISCUSSIONS AND ITEMS FOR A MASTERY TEST

FIRST PANEL DISCUSSION

F Noel, did you ask us to come here this evening to talk about mastery tests?

N A specific mastery test. What we want to try and do is get a prescription, to tighten our ideas as far as possible at this stage, for a mastery test on congruency of plane figures at, perhaps, a Form 4 level.

F Before we start, a friend of mine has suggested that anyone contemplating engaging in writing an objective test should read a book called *The Tyranny of Testing* by Banesh Hoffmann.

N I'll approve of the suggestions, though not of all of Hoffmann's conclusions. His main point though is well taken. It is terribly easy to write questions which are too loose, and admit of more than one, or no, answer. Of course we will make errors. There is no perfect test. All we can hope to do is produce a good test.

R I was just wondering, talking about these objective tests, whether this means we are confining ourselves to some sort of a short-answer or multi-choice form.

N In this particular exercise, yes. We want a test in which a specific answer is unequivocal, so anybody without mathematical training could mark it right or wrong. One comment though: if you can get away with the short answer, this is often preferable to multiple-choice. But when dealing with general verbal understanding, if you want to test fine distinctions of comprehension, then you are almost forced into the multi-choice format. If you leave it open-ended, you never do know, I think, whether they are regurgitating what they've read or heard or whether this is their own carefully thought out idea—whereas five answers, none of which is wrong, but one of which is 'more right' than the others, does force them into this level of distinction. Multiple-

choice questions, then, need not be simple and naive, testing recall, as many people believe. They can be very sophisticated.

F But the things you want us to think about are those which most definitely are of recall.

N That's right.

F Because in a test, for instance in the more experimental matriculation examinations, the whole emphasis is away from the recall of facts—to present a new situation, informative settings. That is, as far as possible asking the student, can you think? Now here we are concerned with just the opposite. Not can you think, but can you remember. And this is terribly important in teaching.

N And we must remember that, as currently constituted, our syllabuses are topic-centred syllabuses, so one can't expect students to problem-solve in this area unless the fundamental factual knowledge is there. For example, in the topic of congruency, there is no hope of solving problems in plane geometry that might involve congruence unless you know under what conditions triangles are, in fact, congruent. It's like trying to play chess without knowing the rules. To some extent then, we are writing a test on mathematical rules.

R Just one point here, why did you choose congruency?

N It seemed to be an important topic for a test in the spatial area. And I suppose I wanted to clarify certain things for myself about the whole nature of congruency. This topic has been taught in various ways and under various guises for four years. What in fact are the end products? What are the performances you expect of students in this area? What would you expect all students to be able to do? Is there some minimum standard of performance and knowledge to expect in this area? That is what I want to try and discover.

R This is my point. Perhaps, under the new syllabus, congruency is going to be rather a slight topic. So might it not be better to take a topic more fundamental to the new courses?

N What spatial area would you suggest?

F Symmetry. For example there is the question of motion geometry which immediately hinges on congruency. Suppose we go back to the classical notion of congruence and say that triangle ABC is congruent to triangle DEF. Then suppose that a transformation occurs and DEF is rotated about the line DF, outside its

original plane. Then we get another triangle DEF which is a mirror image of the old DEF. Then is the last triangle congruent to triangle ABC? It can't be applied to it and fitted to it unless it is turned around outside the plane. In the old syllabus it didn't matter. But in the new syllabus with motion geometry it is an important aspect. And the good youngster will see straight away that rotatory congruency is not the same thing as ordinary congruence.

N Well, then, the topic is perhaps OK. Even if it's an old-hat topic, it is getting a new-hat treatment.

R That's fair enough.

F You see, one goes to three dimensions then, and you might deny all congruence to the elements in a pair of gloves. They can't be superposed. And here the whole question of motion geometry has overridden the classical notion of congruence; and greatly to the improvement generally, I think.

J This brings up the whole question of what you mean by congruency.

N Could we try to become specific now. What we want is essentially a prescription. A prescription for the course which will also be a prescription for our test. We want to prescribe in some detail, not what they have to know, but what they have to be able to do. Because in the test situation this is what we ask them. We ask them to do something.

F Forms 1 to 4?

N It's the end product at Form 4 we are interested in. In terms of congruency, what do we expect the students to be able to do? The doing might involve comprehension. We might ask them to describe or explain certain things, or state the limitations of certain things or the conditions under which certain things happen, or they may have to solve certain problems or prove certain statements. If possible we want a statement in behavioural terms. Not what we are going to teach them, but what the end product is in terms of their behaviour. Because this is what we have to test. Now there may be certain things we might do, certain experiences we might put them through that we don't want to end up in some things they can do. There may be certain attitudes we want to produce and so on, which aren't directly related to a performance. We can't test this. But there must be some cognitive things we want them to do at the end of the course. This is what we must specify.

R Well, if I could put my own opinion here in terms of geometry, I feel that by the time they get to the end of fourth form they know geometrical facts. For example, that the angle at the centre is twice the angle at the circumference. If they could recognize this when they see it. Basically, this is as much as I'd expect.

So if in Form 5 they see a triangle formed by drawing two radii of a circle and they can tell me that the two triangles in that triangle are equal, I'm overjoyed. I don't know if I'm too easily satisfied, but they so often in sixth form overlook the simplest things.

J The fact that they don't see these things in Form 6 doesn't mean that, if they are asked this explicitly, in a test, they won't be able to answer. The problem may be a problem of the context.

R I admit they have to recognize it in the given problem—in, say, mechanics.

N But if it is a specific end product one can teach to it. If one is teaching for this at a mastery level one has a proliferation of visual material, from simple to complex, in all sorts of ways, covered up and disguised, and in all of these situations they have to see the relation involved.

J This needs to permeate through to other topics rather than be treated all at one time. It needs to crop up through the year.

N Agreed, you need to keep on coming back at it. But in the initial learning situation you probably need a far greater number of situations than we have at the moment.

J In terms of congruency then this would imply that the end product is that you want the student to be able to *use the properties* of congruency, however we define congruency, later on—in different spheres, in different areas of work.

F The end product is a failure then as far as university standards are concerned. I'd be overjoyed if 15 per cent of them spotted congruent triangles in a problem which involved this. Even in questions in statics, the difficulty seems to boil down to a basic problem with trigonometry.

J Is it the fact that they've forgotten geometry or the fact that their spatial thinking isn't very good?

F I don't know. But this is geometry in a sense, and congruency in a sense. Because even more fundamental than congruency is similarity. You find this?

R Yes. I was wondering, in fact, just now whether it might not be more useful if we took similarity rather than congruence.

F I would think so. Generally speaking, people aren't interested in congruence. Similarity is a more fundamental thing. Young children are tremendously interested in drawing a picture. The notion of drawing one figure similar to another seems almost born in them. Prehistoric drawings seem to shout this at us. The human race knew about similarity a long time before it contemplated congruency.

J Is this because in our human experience we are used to seeing things at a distance? So everything I look at has this effect of similarity?

N Are you saying that, in terms of what we are saying about mastery, the concept of similarity is more relevant to the educational situation in Forms 1 to 4 than congruence?

F I think it is. It spills over outside the mathematics course into mechanical drawing, into the art courses, and is the foundation for the trigonometry. So it is more basic.

N So shall we switch our test to one on similarity of plane figures?

R Yes.

N So we are interested in the end product in terms of behaviour associated with similarity in Form 4. Is it possible first of all to describe the system we are dealing with in terms of terminology, assumptions, methodology and relationships to real space? Is it possible to state what performance we expect of them in relation to a basic description of the system? Then we can look at the basic properties of the system, in terms of proofs of such relationships, or use of them in limited problem types.

J Would these problems be one-step problems?

N They could probably go up to two-step problems of a specific type. These could still be fitted into a mastery situation. When you get beyond the two-step problem then there are an almost infinite number of complex problems and you are in the 'mathematical learning' level.

J Perhaps this gives some sort of analytic definition of suitable mastery tasks?

N It gives an upper limit, I think.

J Some two-step problems could involve some quite brilliant mathematical deduction, and this is not what you are looking for in a mastery test.

F There is a tremendous intellectual range in this—from boy-scout geometry and measuring to deductive geometry at Form 4 level.

N Well then. If we could work with students up to eight hours per week, if necessary, what sort of performance would we expect most of them to master?

F Some people believe that in third form the tangent is the trig ratio that is most easily understood, but that as soon as you've done the tangent you should bring in the whole six trig ratios at once. They think this gives students a feeling of real confi-dence, and that it's a pity teachers restrict themselves to sin and cos. What do you think?

R This cuts across the idea of restricting your ratios—if you want to call them ratios—to a right-angle triangle.

F Yes, but for youngsters who have been brought up on boy-scout geometry, those ratios are the very thing they need to give them some practical control over situations. To do the theory as most advanced teachers do these days, to think of circular functions out of the blue, is an entirely different approach, and I suppose it's only the brighter youngsters who will see that these are expressions of the same basic ideas.

N Well, would we expect mastery of the six trig ratios in the triangle situation, or not? Are we going to go this far or are we going to stop short of trig ratios?

F It depends on the sophistication of the student. Some regard the idea of the sign of a sine as too abstruse, and like to go back, as it were, to the womb, so a sine is opposite over hypotenuse. Have you found this?

R Well, we've taught, over the past five years, sines and cosines in terms of multipliers instead of ratios. So you can have a negative multiplier. This enables you to go round to 180°. Having done this sort of thing, of course, you find that some-body at the back of the room starts saying something about opposite over hypotenuse.

J It's annoying how some simple things stick in the mind and other more crucial things don't.

R I think though, in terms of similarity, if you're going right through to fourth form, whether you think of it as a multiplier (and perhaps approach it through dilations) or whether you do it from thinking of similar figures, then getting on to ratios, whichever way you teach it, this is something you expect them to know at the end. Talking about dilations, you might start with drawings in Form 1, and by the time they get to Form 4 one of the expressions of this is trigonometry.

F One of the things that should be tested in this mastery test, then, is a thoroughgoing knowledge of trig ratios. And the question now is, how many of them? They must know sine and cos and tan. They must be absolutely perfect in finding them from tables and never add the differences for cos.

R The way out of this, of course, is not to use cos.

F Always take the sine of the complement?

R Yes.

F Well, then, a thoroughgoing knowledge of sine and tan.

J Surely the first question that should be asked in a test of this nature is something like—give two photographs of identical objects and perhaps a third situation, and ask which two photographs show similar figures.

N You mean test the actual concept of similarity?

J Yes.

F Couldn't that be asked at Form 1 level?

J We must still cover him at Form 4. Some students may only be at the Form 1 level still. But can he, at least intuitively, understand what the term similarity implies? This question of trigonometry may enable you to differentiate between the Form 3 and Form 4 student.

N What mathematical concept of similarity do we expect him to have? What are the assumptions or conditions or specifications that define mathematical similarity?

F Would you say the words 'same shape' were mathematical?

N Not to me, but they may be to mathematicians. This seems to me to be a purely intuitive idea.

J You're looking for something more analytic. Something that will press it down like an equation.

F Would you say 'equality of corresponding angles'?

J This fits all right in rectilinear figures. But for people in Form 4, how do you get on with curves? This is perhaps where you need these multiplier and dilation ideas.

R You can bring in the multiplier. There is a fair amount of intuition in it then.

F Most of the children are used to working with cameras, and understand how one photograph can be an enlargement of another. This brings in, without having said so, the notion of a conical projection.

J Why even go to a camera? Why go past the human eye? Equality here for rich and poor!

N Are we in fact almost in the position with similarity as with any sort of system in physics, when you reach a stage where you say—I am the observer and I know what, for example, force is? Do we have to break out of the closed system and say we know what is meant by something?

F Are you going to take similarity then as an undefined notion?

J You could start there but you need this analytic stage as well.

N Are we satisfied with it as an undefined notion in non-rectilinear figures, with its analytic counterpart as well in rectilinear figures? Is this the sort of mastery we expect?

J But we also want to include circles.

F But all circles are similar.

J I'm thinking in terms of circular arcs perhaps combined with linear, so you can get the stained-glass window type of problem.

N What sort of analytic concept would they have with arcs? That the multiplication of the radii is the same factor as the multiplication of the straight side. Is this the only way of getting at this?

R Well, I do it through the dilations all the time. We start by defining what a dilation is and then measurement can give them, if you like, the facts in the early stages. Then simply make use of these facts for recognizing dilation when it's taking place.

N How do you define dilations?

R Simply by a multiplier. So you might have say a dilation of 2 from a given point. This means we join any point to that origin and we double the distance. You are of course assuming that a straight line under dilation will remain a straight line, and so you are assuming invariance. So you can dilate triangles, and the fact that they have equal angles becomes obvious in their measurement as with equal ratios of sides. And they can do the same thing to the circle.

N Should they have mastered this concept of dilation then at a Form 4 level?

F Oh yes, and it should come much earlier in the course.

N Well, we're concerned with end products, so if it is a necessary end product it's really irrelevant whether it's taught in Form 1, 2, 3 or 4.

So we've reached the point where we agree first that the concept of dilation is fundamental and essential to the idea of similarity, and there are various assumptions here about invari-

ance of straight lines and circular arcs under dilation. We would expect them to know that these are assumptions.

F The number of sides in a polygon. That's an invariant too.

J This points out that you can't isolate one particular thing entirely. It must be connected somewhere.

N In terms of methodology there is an implication here as to how the system is described. Do we need to make any comment on the relation of this idea of similarity to real space? Do we expect any differentiation between the mathematical system we've produced for similarity and real bits of string in space on the earth's surface?

F Real bits of string are a model of a real line.

J This is the crux. We're dealing with an analogue.

F I think you could say a model is an imitation of the real thing. That's why my wife complains, 'Don't call me a model.' The three-dimensional thing comes in here in this idea of models.

N Are you saying that the pieces of string are the model for the mathematical system, or the mathematical line is the model for the string on the earth's surface?

F The pieces of string are the model. The straight line is the real thing.

N This is interesting. The physicist would take the opposite view.

J Perhaps we'd better stick to analogue. It's a nice neutral term.

R The practical thing that I would bring in would be the comparison of areas and volumes when you get into solids. In other words, you have a dilation factor of n, so it's n^2 for areas and n^3 for volumes.

N In this test, are we going to stick to plane figures or not? Is the three-dimensional aspect so important that one shouldn't set a test just on similarity of plane figures? As for the average kid, is the solid concept too complicated to expect him to reach mastery?

F We must expect him to master the three-dimensional in Form 4.

J Is it better to introduce the three-dimensional situation and not worry about the plane situation?

N So our title now is 'Similarity of What'?

F Just call it models. Making models.

N 'Similarity of Models.'

F Similarity is irrelevant then. If you have a model it must be a model of something. And the thing that makes it a model is the

existence of similarity. It is similarity that makes the notion of models possible.

N Could we perhaps call it similarity in three and two *dimensions*?

J Somewhere along the line we should bring in something of the idea of generalization. And from two to three dimensions we have this situation.

N How important is this generalization from two to three dimensions and specialization from three to two? Is it something to be stressed in the mastery test or is it more an aspect of higher level problem solving?

J For the mastery test perhaps you are getting at the recognition of the patterns in the dilation equation.

F Recognition of this pattern is the important thing.

R Not quite true. I don't know quite how you teach this unless you can bring it down to some basic rule.

N You expect then definitely to master this square and cubic relationship for areas and volumes of models?

R Yes.

N What other general relationships and conditions would they be expected to know backwards? What theorems, if you like, are essential?

F I don't think they should be able to prove any theorems. Just recognize patterns.

N I was thinking more of the relationship rather than the proof. What theorems would we expect them to know so well as to be almost self-evident?

F Pythagoras.

N Is Pythagoras in similarity?

F Well, the conventional proof involves similarity.

N It comes out of similarity—but would it be part of a mastery test in similarity in three and two dimensions?

F Certainly. Because without Pythagoras you don't know that $\sin^2\theta + \cos^2\theta = 1$. And this is tremendously important. And this is Pythagoras in disguise. For mastery they should know sin, cos and tan, and Pythagoras.

J Perhaps there is a step before this we haven't considered. That's the application of things we discussed previously to particular sets of figures. For example, if we take particular sets of triangles we can work out a set of tests: for example, equivalence of angles is the sort of thing we could test in a mastery test, very directly.

R I think that this is perhaps going right back to the beginning. As well as being able to spot that under a dilation you have a pair of such similar figures, the ratios of which are the dilation factor, you expect them to prove that the angles remain constant.

N Would you expect them to know that if ratios of two sides and included angles are the same, then the figures are similar?

R I don't think so. Though there are occasions where this gives a neater solution to a problem. All angles and all sides seem more reasonable.

F I agree.

N This is too esoteric then?

F So much so that most books refer to it as the ambiguous case.

N Are there, in this area, any proofs which you feel would be essential to reproduce?

R A theorem, you mean?

N Yes.

F I would say not. Though they need to be circumspect. If they talk about similar quadrilaterals they will know that it's not simply the interior angles that are involved but also the angles between diagonals and sides. So there is a good deal of caution involved.

R I agree they shouldn't get the idea that corresponding angles always work.

N Are we limiting similarity to the dilation situation or are we quite happy with mirror images representing similarity?

F Well, the original notion of dilation applies to mirror images whether it's a conical projection or not. They have all sorts of reflections and mirror images. When you spoke of it you were talking of homothetic figures, that is similar figures which are similar and similarly placed in relation to the centre of projection. But you can have reflection, translation, rotation, the lot.

J We didn't really decide how far we should go in discussing the invariance properties of dilation. This is perhaps a keynote to what we've been discussing.

N I put that down as assumptions of the system. Is this OK?

R My point of view is that these sorts of things will be established in the initial stages by drawing in Form 1 and 2. Enough possibly isn't being done, though. This is a discovery approach. But once they've done this I'd be quite happy if they can retain it.

N This is one of the problems in maths for kids of course. It is difficult to specify exactly what it is we *do* want them to retain, and so much is peripheral to the main themes. If they know exactly what it was they had to be able to do and know, the whole task in terms of achievement as measured by mastery tests would be a somewhat simpler one.

R The crucial thing here is time. If you have a month to teach $\sin^2 + \cos^2 = 1$, for example, the better students, the ones you want to keep interested because they will continue with the subject, have it clear after the first week. To do this involves streaming of classes.

N More than that. The educational implications of mastery are quite vast. If you have standardized tests, the very nature of this testing animal presupposes individual rates of progress to a large extent. You can bring groups together at various times for discussion, but largely the progress is based on individual rates. In addition, syllabuses will need to be far more relaxed. A decision must be made. Is it better that this poorer group of students cover the whole syllabus at a 30 or 40 per cent level of mastery, or is it better that they cover half of the syllabus at 85 or 90 per cent of mastery so that at least they get some sort of satisfaction out of the whole procedure? My feeling is that the latter course is a far more satisfactory educational situation. But it's quite impossible to realize it when everyone does the same syllabus, and the length of that syllabus is determined largely by the speed of progress of the upper half. Most secondary school syllabuses involve teachers in a rush job.

PRESCRIPTION FOR A MASTERY TEST ON THE TOPIC 'SIMILARITY IN THREE AND TWO DIMENSIONS'

The following prescription for a mastery test was prepared by the author on the basis of the preceding discussion.

Concept of Dilation

(a) invariance of straight lines, circular arcs and number of sides under dilation
(b) equality of corresponding angles
(c) equal ratios of sides equal to dilation factor M(multiplier)
(d) areas proportional to M^2
(e) volumes proportional to M^3

Concept of Similarity

(a) recognition of pattern
(b) generalization from 2D to 3D
(c) specialization from 3D to 2D

Test for Similarity in Plane Figures

(a) triangles
(b) polygons

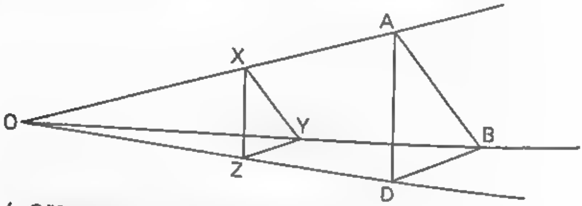
Trig Ratios

(a) tan A, sin A, cos A
(b) $\sin^2\theta + \cos^2\theta = 1$
(c) Pythagoras

FIRST DRAFT OF ITEMS PREPARED FOR A MASTERY TEST ON THE TOPIC 'SIMILARITY IN THREE AND TWO DIMENSIONS'

ORIGINAL ITEMS: SET A

Questions 1 to 9 refer to the following diagram:



OXA, OYB, OZD are coplanar rays
$OA = 3(OX)$ $OD = 3(OZ)$ $OB = 3(OY)$

1 Name an angle equal to $\angle XZY$.

2 What is the magnitude of the ratio $\dfrac{OB}{OY}$?

3 What is the magnitude of the ratio $\dfrac{AD}{XZ}$?

4 List *five* other ratios of sides equal to the ratio $\dfrac{XZ}{AD}$.

5 If $\dfrac{XZ}{ZY} = \dfrac{3}{2}$, what is the magnitude of the ratio $\dfrac{DB}{AD}$?

6 If $\angle XZY$ is a right angle, what trigonometrical function is represented by the ratio $\dfrac{ZY}{XZ}$?

7 If $\angle XZY$ is a right angle, what trigonometrical function is represented by the ratio $\dfrac{DB}{AB}$?

8 What is the magnitude of the ratio $\dfrac{\text{area of } \triangle XYZ}{\text{area of } \triangle ABD}$?

9 If the rays OA, OB, OD are not coplanar, then

A the answers to Questions 1 to 8 would be unchanged.

B the answers to Questions 1 to 8 would all be changed.

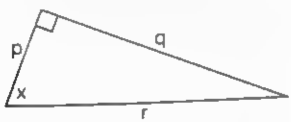
C some of the answers to Questions 1 to 8 would be unchanged, and some would be changed.

D it would not be possible to obtain answers to Questions 1 to 8.

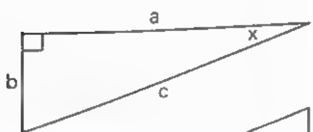
ORIGINAL ITEMS: SET B

1 A tree casts a shadow 10 m long on level ground; at the same time an upright stick, 1 m high, casts a shadow 1 m 25 cm long. How high is the tree?

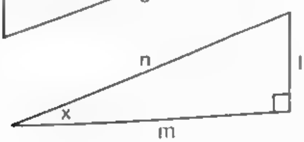
2 x is a number. What circular function (or trigonometrical ratio) of x must be used as a multiplier in each of the following?



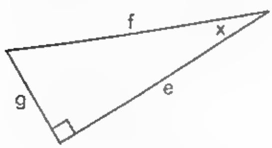
$q = r$.......... x



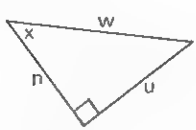
b..........$x = a$



n..........$x = l$



$e = f$..........x

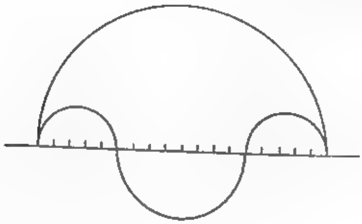


$w = u$..........x

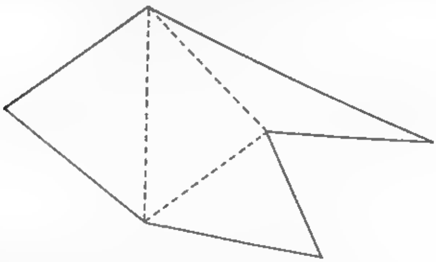
3 Given a pair of compasses and a scale (i.e. a machine-divided straightedge), how many measurements (or direct comparisons) would you need to construct

(i) a *model*, not to any particular scale?

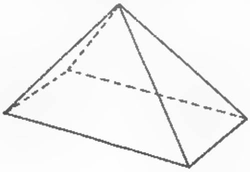
(ii) an exact *copy* of the following diagram? The arcs are all circular.



4 To make a scale drawing *RF* (representative fraction) $\frac{1}{4}$ of the one shown, sides could be measured, and corresponding to them, others constructed, or certain angles could be measured and specified. The least number of sides which could be treated is ...., but then .... angles would need to be copied. The greatest number of sides which could be treated is ...., but then .... angles would need to be copied.



5 To make a *model*, not to any particular scale, of a given piece of material guaranteed to be in the shape of a rectangular pyramid with equal oblique edges, how many measurements (angular and/or linear) would you *need* to make? To make an exact copy (one congruent to the original) how many measurements would you need?



SECOND PANEL DISCUSSION

N There are two basic aspects to discuss about these items: firstly, are these the sorts of things we would expect mastery of at Form 4 level? Secondly, can they be improved? Are they valid, and are they clear and precise?

F About the information. Should we delete 3 (under the first diagram in set A) and replace it with k?

R This makes it a bit more difficult.

N Yes, I wanted it to be as simple as possible.

F It may be logically simpler but it is psychologically more difficult because the diagram is not drawn to scale.

N Perhaps the solution is to make a better diagram.

R I think so.

J Now, as I understand it, this test is to test the whole range of mastery from Form 1 to Form 4. Shouldn't we first check that they know what is meant by similar figures? Some Form 3 children just mightn't be able to do this question and we want to know why.

N This is true. Some teachers may not be teaching the topic in this way. Nevertheless, if these are the fundamental aspects, the basic concepts, we should make this the test. The teaching will follow if it is accepted that these are the basic areas for mastery.

J I just feel there should be a question one stage prior to this area at a more elementary level, simply testing whether they know what is meant by similar figures.

N Have you got one then?

J Just a rough idea. You put down successive diagrams and ask them to draw lines to do mapping on similar figures. Doing this you can test whether they can differentiate things which are sometimes called similar from those which are mathematically similar.

F I think that is good. It takes us right back to the basis of similarity in prehistory. It's devoid of vocabulary.

R I'd agree. On a different line, but somewhat similar, should we ask them something about the transformation? Given this information on the diagram, ask what transformation would produce triangle ADB from triangle XYZ? Not that I'm an advocate of labelling things; but if they can say 'this is a dilation', then they are thinking along the right lines.

N This could be written as a multiple-choice, perhaps?

R This would be good. They would just have to print the right word rather than memorize it.

F At Form 3 level verbalization is required, so I would think mastery involves the actual knowing and recall of the words, so I would not approve the multiple-choice.

N What question would you ask?

F What kind of transformation or mapping (whichever word you want to use) would produce triangle ABD from triangle XYZ?

N Would you accept only dilation? Or is enlargement OK?

F Either of those. Even projection might be acceptable.

N This is one of the problems of leaving it open-ended. In multiple-choice format you can specify the various types of transformation in their normally acceptable mathematical usage.

J Coming back to this point of difficulty, I think we must cater for those who are not at Form 4 level. So with terminology, perhaps the multiple-choice might be better.

N We could come at this from the other direction and ask which one of a set of diagrams represents a dilation. Going from verbal to figural rather than the other way round.

F Perhaps someone could prepare some items along these lines and we will discuss them at the next meeting.

N Well, would someone like to comment on Question 1?

R Are we going to be very fussy about ideas of angles equal to, or should we say equal in measure to?

F Some teachers are fussy about differences between measure and magnitude and the like. It would be a pity to do this against their interest.

N 'Name an angle equal in measure to.' How's that?

R Good.

N Can angles be equal other than equal in measure? Can the statement mean anything else?

R Well I think they are trying to get away from saying two angles are equal when they're virtually the same angle.

N An angle is envisaged as something other than its measure?

F An angle really, according to some people, is a set of points, a union of two rays. I prefer not to have to answer the question.

R That's the trouble. Why should we create such a fuss about this in Forms 1 and 2, about questions that are very deep?

J Depends on the teacher, doesn't it. If the teacher isn't too happy about it, he'd be better not to do it perhaps.

R You're tending to do something because it's thought to be rigorous. But perhaps you're doing just the opposite.

F Perhaps the more the teacher knows about this, the more flexible he is likely to be.

N Let's leave it as measure then we won't upset anybody.

J Isn't it marvellous how, when you're setting these papers, though, this is what develops? You learn such a lot yourself.

F Rather. We all learn from this sort of discussion.

R Question 2 now: you say what is the magnitude of the ratio. Now ratio is a number so it can't have a magnitude.

N 'What is the value of the ratio?' Is that right? Question 3, 4 and 5 need to be changed in the same way.

R How do you score number 4?

N Perhaps 2 marks for 5 correct answers and 1 mark for three or 4 correct answers?

F Yes, it is the sustained effort that should bring rewards.

J Is there any advantage in not giving the number? Let them look for all the possible ratios.

N Another way is to ask them to list two other ratios. Then ask how many such ratios are there.

F Ask two questions out of it.

N Yes.

F I'd be agreeable to that.

J It could have advantages from the point of view of marking.

R There would also tend to be less careless errors if handled this way. We don't want to penalize transcription errors.

N What about Questions 6 and 7?

R These are like the questions in Set B. I don't like this linking up of trigonometrical *functions* with ratios. Unless we are trying to approach trigonometry through two aspects, there could be confusion.

F I prefer the more straightforward approach in Set B.

R Question 8 gives us area. But then I'd like to add a further question about comparison of volumes of pyramids *OABD* and *OXYZ*.

F Excellent.

N Looking back now at the eight questions we've accepted, are these the sorts of questions we want to ask? Are they basic and fundamental enough for a mastery test?

F No school boy of 300 BC should have been unable to answer

those. That's pretty fundamental.

J Before we leave this, should we have a question linking up the initial question of the concept of similarity with this group of questions? A question to bridge the gap between the physical situation and the mathematical definition?

N Could one ask a verbal question about why this is a dilation? Or could we produce four sets of diagrams, some of which are mathematically similar figures, some not?

F Mathematics is a game, and the word similar does have quite different connotations in mathematics to what it has in everyday usage. Here it has a very precise meaning, and we need to tie this meaning down.

J Yes, we need a question to see if they can separate the colloquial meaning of similar from its mathematical meaning.

F Such a question should be devised.

N Perhaps you could try it, John. It might be possible to do it either through alternative diagrams or through alternative verbal descriptions.

J I'll try. Let's go on to Set B now.

F Question 1 is a piece of straight-out boy-scout geometry that any Form 1 student should be able to do.

N This is still part of the mastery we need at Form 4.

R You mention the ground is level and the stick upright. How about the tree?

F It doesn't matter.

R Height will be taken as vertical height?

F Yes.

R Perhaps leave out upright so that there isn't the suspicion of a trick.

N I'd prefer to have an upright tree—to tighten rather than loosen it.

J This question isn't anything new. This is a little naive in a sense. It isn't really practical. It works for tall trees, but not for short bushy ones.

F Perhaps we should make it a flag pole.

J This solves one problem. But it still isn't the sort of imaginative thing the child might want to measure. Let's get something not so old-hat, and more practical.

R High buildings have problems.

N You might measure the height of an edge rather than the total

height, because of masking.

J Something like the dome of St. Paul's Cathedral.

F Or a TV transmitting antenna.

N The bother with the dome of St. Paul's is that some kids will know the height.

J Perhaps it should be a check measure. Could you check the height of the Twickenham goal posts?

N We could give what the height of the goal posts is supposed to be. Then give information, and ask them to calculate the height.

J It could be more practical and done by line of sight rather than shadows.

N Do we want it verbal or do we want a pictorial representation? Would the question ever appear in verbal form except in an exam?

J Perhaps we just ask how they would estimate the height.

R That's a bit wide open. They might drop a string from the top.

N Perhaps they could be given a surplus of information. Give too many measurements. Ask which measurements they would need to calculate the height.

R How is this different from the similar triangle questions asked in Set A?

N Mathematically it may be very similar, but the situation in which the problem is presented is much different, and this makes it a quite different problem for the student.

R This is meant to be a problem then?

N A type with which they are familiar.

F I like this idea of asking what measurements are necessary.

N Then we could also go on to ask them to calculate the height.

R If they can't answer the first question, then they won't be able to answer the second.

N Theoretically, but not in practice. Test items don't work that way. Sometimes the easy question is missed and the difficult one is correct. Perhaps you could rewrite that one in two parts then. Now what about Question 2?

F This kind of thing can produce initial bafflement but I'm sure students can master it.

R Let's call it trigonometrical function, not ratio.

N Perhaps you could solve the problem by giving an example. If the kid is not sure what a circular function is, but you mention cos or sin or tan to key him off, then he knows exactly what

you mean. The word might stop them, but given a bit of help they might get the five right.

J Could a diagram be given showing the unit circle with projections on the axes and the student asked to write the function represented by a particular segment? This would do the same sort of job.

F A piece of charity.

N We must remember this whole test could be described as 'charity'. Our expectation is for 90 per cent. If we had the right diagnostic tests they wouldn't do this test unless they were almost certain to get 90 per cent.

J This is surely a fundamental step which it is necessary to test.

F Certainly.

R What about this last one? Should they have mastery of the whole six functions?

N For mastery, I think we decided on sin, tan, cos only. That's the way I interpreted the last meeting. But we could write it with the function on the bottom of the fraction.

R One final point about the use of x. Are most candidates going to be familiar with the use of $\sin x$?

F Would you prefer A?

N What is A now?

F A is the number which is a measure of the angle. A is not a point. Is this confusing?

R Perhaps an A inside the triangle with an arc in it?

F Don't put an arc around it or they'll think A stands for an angle. A is a number.

N Do we write then that A is a number which is a measure of the angle.

F No. It's a number, full stop.

R In terms of the diagram, perhaps we should say that A is a number less than $\frac{\pi}{2}$

N Surely the number must be related to the diagram? And how is it related to the diagram if it isn't a measure of the angle?

R This is the problem.We can start off with definitions. But as soon as we relate them to this diagram we are bound to relate it to angles.

J Perhaps we could say A is a number associated with the angle marked on the diagrams.

N Is this right? This is a practical situation. The mathematical model you are going to use to solve this is one involving circular functions. So you must then associate the circular function in your model with a measurement in this diagram, and you don't want to say that the circular function is the measurement of the angle but that there is a one-to-one relationship between them. Is this the situation?

F Yes. But you can't put all that into the question. If you put A in the diagram, you don't have to explain and all should be well. They know the rules of the game.

R One thing I'm not clear about myself in the teaching of this—and teaching and testing are interrelated—is how one does make the link between the idea of the circular function of a number, and the looking up of the sine of 30° from a set of tables.

F That's all right. Degree itself is a number. Degree means $\frac{\pi}{180}$ Just a number. So 30° is 30 times this number.

J The point is whether we should be labouring this functional aspect at this stage, when books of tables are so basic.

N We haven't yet written any items to test their skill in looking up tables. These are obviously necessary.

F Let's go on to number 3 now.

N I can't do this.

J Is it more the sort of question for an achievement test rather than a mastery test?

N I'm not sure of the processes here. When you say how many measurements to construct that large semi-circle, I'm not sure what this means.

F The answer is none for a model. All semi-circles are similar. For a copy any single measure (of say radius or diameter) will define the size of the semi-circle.

N If these are the two points they got at, perhaps the answer could be asked more directly. It seems very subtle as set here.

J The student may not be too clear about exactly what is meant by measurement.

R Perhaps the way you've phrased Question 4 is a neater way of approaching it. You're getting back to the figure instead of talking about measurements.

J That comment about all semi-circles being similar is the sort of thing I'd envisage in those very early questions.

F Perhaps you could draw two different shapes of ellipses for the case where they're not similar?

N Which one or more of the following is not true?

A All equilateral triangles are similar.
B All right-angled triangles are similar.
C All ellipses are similar.

J I've got something like that here. In which of the following sets are the elements all similar:

the set of circles
the set of rectangles
the set of houses
the set of cubes
the set of people
the set of right-angled triangles?

R Isosceles triangles?

J This is more verbal, of course.

R Perhaps this question is closer to what we want for a mastery test.

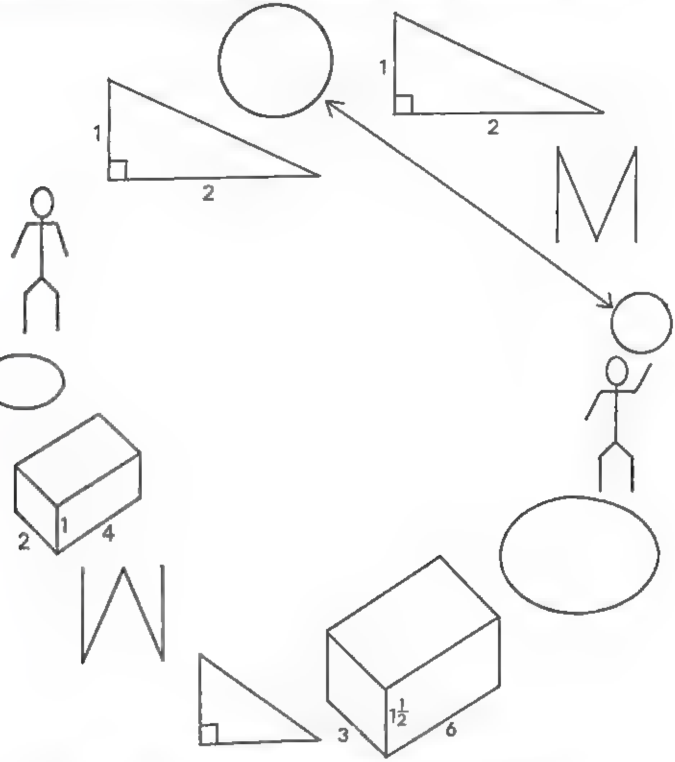
N Could we have a second part to this? If you knew it was a semicircle, how many measurements would you need to take? This question could be asked also.

F Could we stop now for a cup of tea?

N
R Yes.
J

CORRECTED AND NEW ITEMS FOR A MASTERY TEST ON THE TOPIC 'SIMILARITY IN THREE AND TWO DIMENSIONS'

1 Study the following sets of objects. Then, draw a single line connecting pairs of figures which appear to be similar. One such line (between the two circles) has already been drawn for you.

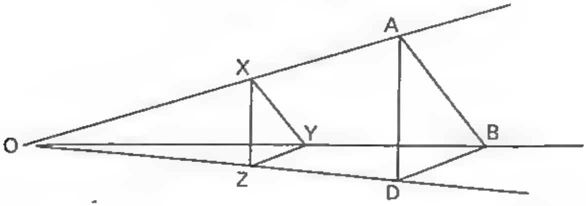


Attempting to estimate ability to recognize similarity whilst varying orientation, measurement, dimension, etc.

2 For which one or more of the following sets is the relation 'is similar to' valid for all pairs of elements?

A the set of semicircles
B the set of rectangles
C the set of houses
D the set of cubes
E the set of people
F the set of right-angled triangles
G the set of isosceles triangles
H the set of congruent triangles

Questions 3 to 11 refer to the following diagram:



OXA, OYB, OZD are coplanar rays
$OA = 3(OX) \qquad OD = 3(OZ) \qquad OB = 3(OY)$

3 Name an angle equal in measure to $\angle XZY$.

4 What is the value of the ratio $\dfrac{OB}{OY}$?

5 What is the value of the ratio $\dfrac{AD}{XZ}$?

6 List two other ratios of sides equal to the ratio $\dfrac{XZ}{AD}$.

7 How many ratios of sides, each equal to $\dfrac{XZ}{AD}$, can be found?

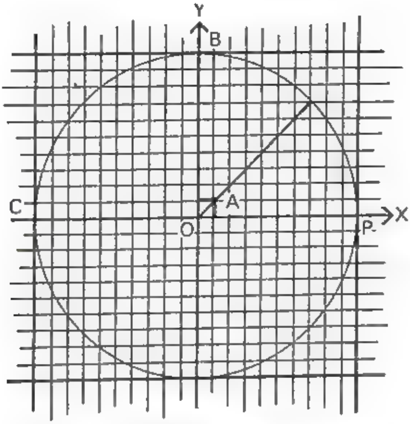
8 If $\dfrac{XZ}{ZY} = \dfrac{3}{2}$, what is the value of the ratio $\dfrac{DB}{AD}$?

9 What is the value of the ratio $\dfrac{\text{area of } \triangle XYZ}{\text{area of } \triangle ABD}$?

10 If the rays OA, OB, OD are not coplanar, then
A the answers to Questions 1 to 8 would be unchanged.
B the answers to Questions 1 to 8 would all be changed.
C some of the answers to Questions 1 to 8 would be unchanged, and some would be changed.
D it would not be possible to obtain answers to Questions 1 to 8.

11 What is the value of the ratio
$$\frac{\text{volume of pyramid } OXYZ}{\text{volume of pyramid } OABD}\ ?$$

Questions 12 to 14 refer to the following diagram:



OX and *OY* are perpendicular axes and *PBC* is a unit circle

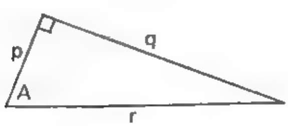
12 What is the value of sin A?

13 What is the value of cos A?

14 What is the value of tan A?

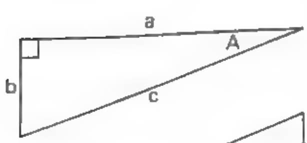
Questions 15 to 18

A is a number associated with the angle marked on the diagrams. What circular function (or trigonometrical ratio) of A must be used as a multiplier in each of the following? The first example has been completed for you.



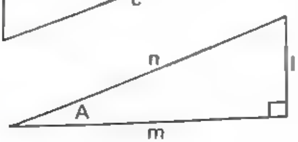
$q = r$ (sin A)

15



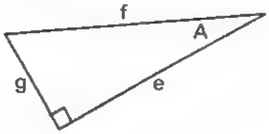
b (............ A) $= a$

16



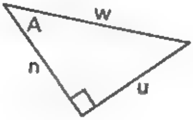
n (............ A) $= l$

17



$e = f\ (\ldots\ldots\ A)$

18

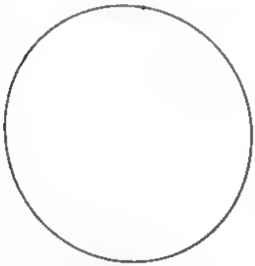


$$w = \frac{u}{\ldots\ldots\ A}$$

Questions 19 to 21

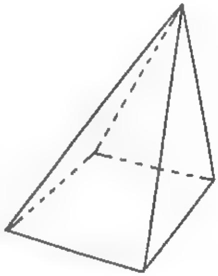
Each of Questions 19 to 21 illustrates a geometrical shape. In each case you are told the name of the shape. You are to specify how many measurements you would need to make of each shape in order to construct a similar shape.

19



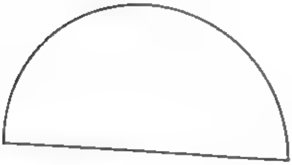
sphere

21



square-based right pyramid

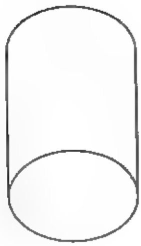
20



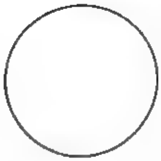
semi-circle

Questions 22 to 24

Each of Questions 22 to 24 illustrates a geometrical shape. In each case you are told the name of the shape. You are to specify how many measurements you would need to make of each shape in order to construct an exact replica of it.

22 

cylinder

24 

circle

23 

isosceles triangle

APPENDIX II

TEST ITEMS FOR TESTS OF MATHEMATICAL LEARNING

Major Categories

I Analysis in a *structured system:* Where the combination of data and questions is such as to indicate relatively clearly the way to a solution.

II Analysis in an *unstructured system:* Where there is little or no clue in the question of data regarding the method of solution.

III Read and comprehend *new mathematical systems:* Where there is emphasis on the development and comprehension of a well defined structure.

IV Follow and construct a *proof.*

Sub-categories

A *Verbal:* The analysis required of the material in the *data* is mainly an analysis of verbally expressed relationships.

B *Tabular:* The analysis required of the material in the data is mainly an analysis of data in tabular form.

C *Graphical:* The analysis required of the material in the data is mainly the analysis of the graph.

D *Figural:* The analysis required of the material in the data is mainly an analysis of figural relationships.

E *Symbolic:* The analysis required of the material in the data is mainly an analysis of symbolic relationships.

F *Numerical:* The analysis required of the material in the data is mainly an analysis of numerical relationships.

Difficulties Difficulties are indicated by a circled number, thus ③. The relative difficulties are for Form 4 students based on results on the Commonwealth Secondary Scholarships Examinations. This group may be considered to contain the most able 70 per cent of the secondary school population.

Difficulty	*Percentage of students giving correct answer*
①	More than 70 per cent (easy)
②	Between 56 and 69 per cent
③	Between 45 and 55 per cent
④	Between 31 and 44 per cent
⑤	Less than 30 per cent (hard)

I ANALYSIS IN A STRUCTURED SYSTEM

A VERBAL

1 John is three times as old as Peter. In two years' time he will be twice as old as Peter. Peter's present age, in years, is ①

A 3
B 5
C 7
D 9
E none of these

Questions 2–5

Read the passage below carefully and then statements 2–5 following. These statements are all true conclusions based on the passage. For each conclusion state the letter of the sentence in the passage which must be used to reach it.

(K) In the space flight of a rocket a factor called the mass-ratio is most important.
(L) The take-off mass of any rocket is the sum of three quantities called the pay load, dead weight, and fuel load.
(M) The pay load is the mass of the compartments used for carrying explosives, scientific instruments and controls, or pilot and crew, and their contents.
(N) The dead weight is the total mass of the rocket structure plus all motors, fuel pumps and such like; and the fuel load is the mass of the fuel carried.
(O) After the fuel has been used up, the pay load and dead weight form the remaining mass.
(P) The mass-ratio is the ratio of the take-off mass to the remaining mass.
(Q) Thus, if the take-off mass is 45 tonnes and the remaining mass is 15 tonnes, the mass-ratio is 3 : 1.
(R) The higher the mass-ratio, the greater the fuel load and the greater the range of the rocket.
(S) When a single-stage rocket is fired, it travels less than its own length during the first second.
(T) However, the velocity of a single-stage rocket increases rapidly, and it is about 13 m/sec at the end of the first second,

26 m/sec at the end of the second, 39 m/sec at the end of the third, and so on.

(U) Soon the velocities become very large, partly because the rocket is losing weight steadily as its fuel is consumed and partly because the effect of the motor increases as the rocket reaches the thinner layers of atmosphere where there is less resistance to its motion.

2 Half a minute after take-off a single-stage rocket has a velocity of the order of 400 m/sec. ①

3 If two rockets have the same dimensions and take-off mass, but one has a mass-ratio of 4 : 1 and the other a mass-ratio of 3 : 1, then the rocket with the 4 : 1 ratio will outdistance the other. ①

4 If improvements in scientific instruments result in lighter and smaller instruments, it will be possible to use more instruments for a given pay load. ①

5 If in a rocket the fuel load accounts for more than half of the take-off mass, then the fuel load of this rocket exceeds the pay load and dead weight. ④

Questions 6–8

A man who wishes to give up smoking enlists the aid of a friend who agrees to sell him cigarettes charging 2p each on the first day, 4p each on the second day, 8p each on the third day, and so on (i.e., doubling the charge on each successive day). The smoker buys cigarettes only from his friend, and is unable to spend more than £2 on any one day for cigarettes.

6 Which one of the following is the first day on which he is unable to buy a cigarette? (He may not buy fractions of a cigarette.) ①

A the fiftieth day
B the twenty-first day
C the fifteenth day
D the eighth day
E the sixth day

7 How many cigarettes can he buy on the fifth day? ②

A 1
B 6
C 7
D 10
E 16
F more than 16

8 If he buys all the cigarettes he can afford for the first 3 days, what is the average price in pence that he pays per cigarette over these 3 days? ⑤

A less than 4
B exactly 4
C between 4 and 5
D exactly 5
E more than 5

9 This question concerns negative numbers. One of the statements given is false. Select it. ③

A The result of the addition of two negative numbers is always a negative number.
B The result of the subtraction of two negative numbers is always a negative number.
C The result of the multiplication of two negative numbers is never a negative number.
D The result of the division of one negative number by another is never a negative number.

Questions 10, 11

A man lost in the Sahara desert walked according to the following pattern (□ represents a missing number):

5 km north, 3 km east, 2 km south,
1 km west, □ km north, 5 km west,
7 km south, and finally, 2 km east,

at which point he could go no further, and was then located due west of his starting point.

10 What number does the □ represent? ③

11 How far was the man located from his starting point? ③

12 The number of years in Emily's age is divisible by itself and exactly four other whole numbers greater than 1. In two years' time it will again be divisible by exactly four whole numbers excluding itself and one. Which one of the following could be Emily's present age in years? ③

A 12
B 18
C 28
D 32

13 The least number of things which may be lost out of p dozen things so that the remainder may **not** be grouped in lots of three is ③

A 1
B 2
C $4p - 1$
D $4p$

14 Temperatures on the Fahrenheit scale (R) are obtained by multiplying the corresponding Centigrade scale temperature (S) by 9, dividing this answer by 5, and then adding 32 to the result. The formula so obtained can be rearranged to give an expression for S in terms of R. Which one of the following correctly expresses this transformation? ③

A $S = \dfrac{5R - 32}{9}$

B $S = \dfrac{5(R - 160)}{9}$

C $S = \dfrac{5}{9}R - 160$

D $S = \dfrac{5}{9}R - 32$

E $S = \dfrac{5(R - 32)}{9}$

15 Poultry food costs £20 per tonne (1 tonne = 1 000 kg). Which one of the following is the average cost per day of food for 160 hens if each hen eats an average of 250 g of poultry food per day? ③

A £0.80
B £1.00
C £2.50
D £5.00
E £7.50

Questions 16, 17

There are several temperature scales which have been used in scientific work. Two of these are the Kelvin scale and the Reaumur scale. 273° on the Kelvin scale is almost equivalent to 0° Reaumur (freezing point of water) and 373° is almost equivalent to 80° Reaumur (boiling point of water). Between the two 'fixed' points (i.e., the freezing point of water and the boiling point of water), the Reaumur temperature scale is divided into 80 equal intervals each of size 1 Reaumur degree. Similarly, the Kelvin scale is divided into equal intervals each of size 1 Kelvin degree.

16 Which one of the following changes in temperature on the Reaumur scale would correspond to a change of 10° on the Kelvin scale? ③

A 4°
B 8°
C 10°
D $12\frac{1}{2}$°
E 16°
F 30°

17 The temperature on a sunny day is 20° Reaumur. What would the temperature reading be on the Kelvin scale? ④

A 24°
B 25°
C 40°
D 100°
E 293°
F 298°

18 The story is told of a man who was cursed, and the curse was extended to his sons and their sons unto the seventh generation. Now he had ten sons, and they each had ten sons, and they each had ten sons, and so on unto the seventh generation. Assuming his sons are counted as the first generation, the total number of people cursed is approximately ④

A the population of Great Britain.

B the population of London.

C the population of the USA.

D the population of Asia.

E the population of Brighton.

F 71.

19 To build a fence a contractor charges £2 per metre plus £1 for each post. If the distance between the centres of adjacent posts is 3 m, how much does he charge to build a straight fence 18 m long? ④

A \$54

B £39

C £41

D £42

E £43

20 The average weight of eight members of a rowing crew is 71 kg. When the weight of the cox is added in, the average weight of the crew and cox becomes 68 kg. ④
The weight of the cox is

A 43 kg

B 44 kg

C 49 kg

D 52 kg

E none of these

21 A girl has n cubes, all of the same size, from which she tries to build a big cube. Nearly finished, she finds that she does not have enough cubes to complete the building of the big cube; just one row along one edge is missing. If the big cube needs 216 small cubes to build, how many cubes has the girl? ④

A 6

B 144

C 162

D 200

E 210

F $216 - n$

22 The average weekly wage for adult males in Scotland at April 1973 was £36.01. The average wage for adult females was stated as equal to the wage for adult males less 25 per cent. Which one of the following gives the annual wage of a female worker paid at the average female wage rate? ④

A $£\left[52\left(\frac{3\,601}{100} - \frac{25}{100}\right)\right]$

B $£\left(52 \times \frac{3\,601}{100} \times \frac{100}{125}\right)$

C $£\left(52 \times \frac{3\,601}{100} \times \frac{75}{100}\right)$

D $£\left[\left(52 \times \frac{3\,601}{100}\right) \div \frac{100}{125}\right]$

E $£\left[\frac{3\,601}{100}\left(52 - \frac{25}{100}\right)\right]$

23 The pilot of a jet fighter must be conscious of the time his eyes can be closed when he blinks quite normally, as it amounts to a short blackout period. On an average this time is 0.3 seconds. State which of the following is *closest* to the distance that he would have moved during one of these normal 'blackouts' when he is travelling at 2 000 km/hour. NB 36 km/hour is the same speed as 10 metres/second. ④

A 2 m

B 15 m

C 150 m

D 1 km

E 6 km

Questions 24–28

Each of the following questions consists of an initial section containing some information and a question. The information is not sufficient to answer the question. Following this initial section there are two additional pieces of information (1 and 2). Consider these carefully and then circle on your answer sheet

A if statement (1) *alone* is sufficient additional information to answer the problem but statement (2) alone is not sufficient.

B if statement (2) *alone* is sufficient additional information to answer the problem but statement (1) alone is not sufficient.

C if both statements (1) and (2) *together* are sufficient additional information to answer the problem but neither alone is sufficient.

D if each of statements (1) and (2) is sufficient additional information *by itself* to answer the problem.

E if both statements (1) and (2) *together* are not sufficient additional information to answer the problem, and further information is needed.

24 When a body slides over a rough surface there is a reaction force, N units, perpendicular to the surface, and a friction force, F units, along the surface. These forces are related by the equation $F = \mu N$ where μ is a positive constant. Is the force F larger than the force N? ⑤
(1) The value of μ varies from one situation to another.
(2) μ is less than 1.

25 In a certain Austrian town Italian or German or both are spoken by the entire population. Only 75 per cent of the population speaks Italian. What percentage of the population speaks both Italian and German? ⑤
(1) Only 83 per cent of the population speaks German.
(2) Only 17 per cent of the population cannot speak German.

26 At the beginning of one year there are 250 boys and 200 girls in a school. At the beginning of the next year has the number of girls decreased by 25 per cent? ④
(1) The number of children in the second year is the same as in the first year.
(2) At the beginning of the second year there are 300 boys in the school.

27 ABC is any triangle. Is the side BC shorter than the side AB? ③
(1) $\angle CAB = 60°$ (2) $AB = 2$ cm

28 The following is a set of five numbers, one of which is out of place: 2, 5, 10, 15, 26. Which number is out of place? ④
(1) The remaining four numbers have factors which are prime numbers.
(2) The remaining four numbers are each one greater than a perfect square.

29 Consider the four statements below concerning numbers which may be negative or positive, and select the **two** that are true. ⑤
A If two numbers are equal then their squares must be equal.
B If the squares of two numbers are equal then the numbers themselves must be equal.
C If two numbers are not equal then their squares cannot be equal.
D If the squares of two numbers are not equal then the numbers themselves cannot be equal.

30 A shopkeeper increased his original stock by 20 per cent and then sold 20 per cent of all his stock. When he checked his stock against the original stock, he would expect ⑤
A no difference.
B 4 per cent decrease.
C 4 per cent increase.
D 5 per cent decrease.
E 5 per cent increase.

31 Following an outbreak of an epidemic in the town of Schmerz, two doctors made initial and independent recommendations about the area that should be quarantined. One week later they both decided to change their recommendation. ⑤

Doctor X initial recommendation—no person be allowed to move more than 10 km from the post office (area 1).

Amended recommendation—no person be allowed to move more than 5 km from the post office (area 2).

Doctor Y initial recommendation—all persons be restricted within an area of 300 square kilometres (area 3).

Amended recommendation—all persons be restricted within an area of 150 square kilometres (area 4).

With respect to the four areas it is correct to say that

A area 1 is greater than area 3 and area 2 is greater than area 4.
B area 1 is greater than area 3 but area 2 is less than area 4.
C area 1 is less than area 3 but area 2 is greater than area 4.
D area 1 is less than area 3 and area 2 is less than area 4.

32 A girl aged seventeen is left £x in her grandmother's will. The money is put in trust for four years until she turns 21. If the money invested earns interest at 5 per cent per annum calculated on the original capital (i.e. simple interest), then the girl will eventually receive ⑤

A $\dfrac{£6x}{5}$
B £$(x + 20)$
C £$\dfrac{x}{5}$
D £$5x$
E none of these

33 A rectangular lawn, x m long and y m wide, is surrounded by a garden bed such that each point of its outer edge is p m distant from the nearest part of the lawn. The total length in metres of the outer perimeter of the garden bed is ⑤

A $2(x + y)$
B $2(x + y + 4p)$
C $4x + 4y + \pi p$
D $2(2x + 2y + \pi p)$
E $2(x + y + \pi p)$

B TABULAR

Questions 34–36

In a certain election there are five candidates P, Q, R, S, T of whom two are to be elected. Voters numbered their papers 1 to 5 in order of preference, and all voting papers were found to be satisfactory. The scrutineers in this election prepared the following table (leaving certain gaps):

CANDIDATES	VOTES 1st	2nd	3rd	4th	5th
P	3	8	2	5	2
Q	2		3		7
R	7	5	3	3	2
S	5	2		4	4
T	3	1	7	4	5

A candidate is automatically elected if he receives 35 per cent or more of the first votes. If less than two candidates are automatically elected, then the other candidates are arranged in order of preference by awarding 5 points to a candidate for each first vote he obtains, 4 points for each second vote, 3 points for each third vote, and so on, and totalling the points so awarded.

34 The number of second votes received by candidate *Q* is ②

A 2
B 4
C 5
D 8
E 20

35 Which one of the following statements follows from the information given? ②

A Only candidate *P* was automatically elected.
B Only candidate *S* was automatically elected.
C Only candidate *R* was automatically elected.
D Both candidates *R* and *S* were automatically elected.
E No candidate was automatically elected.

36 The two candidates finally elected were ②

A candidates *P* and *R*.
B candidates *R* and *T*.
C candidates *R* and *S*.
D candidates *S* and *T*.
E candidates *P* and *S*.

Questions 37–40 refer to the following table and information:

The following statistics represent the results of an examination of 900 candidates. The examination comprised 12 questions, each of which was worth a maximum of two marks.

Question No.	1	2	3	4	5	6	7	8	9	10	11	12
N	897	688	705	722	820	692	600	693	703	796	467	402
B	562	328	172	45	335	154	60	54	543	348	82	8
Z	20	194	143	326	268	111	270	121	8	38	150	325

N is the number of candidates attempting a question.
B is the number of candidates achieving full marks on a question.
Z is the number of candidates achieving zero marks on a question.

37 What percentage of candidates attempted Question 7? ①

A 33⅓ per cent
B 10 per cent
C 55 per cent
D 66⅔ per cent

38 What percentage of candidates who attempted Question 7 obtained full marks? ②

A 6⅔ per cent
B 10 per cent
C 15 per cent
D 45 per cent

39 How many candidates obtained full marks on both Question 7 and Question 8? ④

A 600
B 693
C 114
D 14
E There is not sufficient information to tell.

40 Which question was answered most successfully by those who did it? ③

A 1
B 2
C 9
D 12

Questions 41–43

The table gives numerical ratings of 10 articles on three characteristics.

Article	*Characteristic* (i)	*Characteristic* (ii)	*Characteristic* (iii)
A	5	16	9
B	12	8	3
C	3	7	16
D	2	14	5
E	17	5	7
F	9	15	18
G	6	9	10
H	11	7	4
I	20	12	6
J	7	10	15

It is possible to make an ordered selection from the articles for a given purpose by applying a fixed rule to the numerical information. For example, if we selected according to the rating on characteristic (ii) we would take the articles in the following order A, F, D, I, J, etc.

In each of the questions the order of selection for the first four articles, using a fixed rule, is shown. In each case you are to find the next article that would be chosen.

41 D, C, A, G, . . . (hint: based on a single characteristic) ①

42 F, C, J, G, . . . (hint: based on a single characteristic) ②

43 F, I, J, A, . . .(hint: based on all three characteristics) ④

Questions 44–48 refer to the following information:

The decimal system is the normal system of counting in tens.
The tertiary system is a system of counting in threes.
The octal system is a system of counting in eights.
The table below shows some equivalent numbers expressed in each of the three systems.

Decimal	*Tertiary*	*Octal*
0	0	0
1	1	1
2	2	2
3	10	3
4	11	4
5	12	5
6	20	6
7	21	7
8	22	10
9	100	11
10	101	12

44 Write the *decimal number* 15 in the tertiary system. ④

A 5
B 120
C 122
D 113
E 300

45 Write the *decimal number* 16 in the octal system. ③

A 2
B 18
C 24
D 20
E 121

46 Write the *tertiary number* 222 in the decimal system. ④

A 26
B 22
C 20
D 74
E 222

47 Write the *tertiary number* 201 in the octal system. ②

A 15
B 20
C 201
D 209
E 23

48 Write the *octal number* 17 in the decimal system. ②

A 13
B 15
C 17
D 19
E 21

Questions 49–53

Index of Industrial Production 1957 = 100

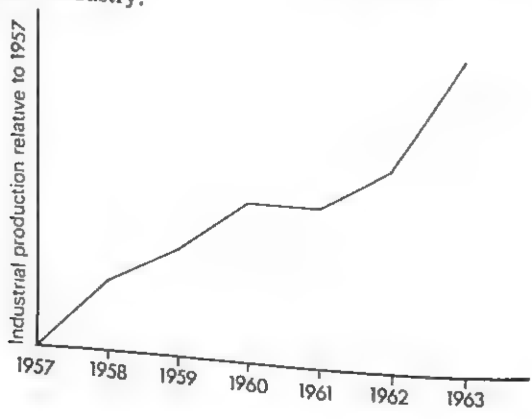
Period	Total	Wood industry	Paper industry	Metal industry	Textile industry	Other industries
1957	100	100	100	100	100	100
1958	111	99	115	111	109	111
1959	114	78	122	113	110	116
1960	117	82	134	115	116	117
1961	113	87	134	107	113	109
1962	123	98	144	120	125	120
1963	140	122	167	140	143	132

The above table shows the development of industrial production in a certain country over the period 1957–1963. The year 1957 was taken as the base for this table (100 units of production), and the production in subsequent years was expressed in terms of this base. For example, in the wood industry the production in 1959 was 78 per cent of the 1957 production.

Use this table to answer the questions following.

49 The graph depicts the development of industrial production in the ①

A wood industry.
B textile industry.
C metal industry.
D paper industry.
E other industries.



50 The industry whose industrial production advanced most rapidly in any one year of the 1957–63 period was the ③

A wood industry.
B paper industry.
C metal industry.
D textile industry.

51 The increase in the total industrial production in the 1957–63 period was ①

A 22 per cent.
B 32 per cent.
C 40 per cent.
D 43 per cent.
E 67 per cent.

52 The smallest positive increase, relative to the previous year's production in the total industrial production, occurred in ⑤

A 1958.
B 1959.
C 1960.
D 1962.
E 1963.

53 **Suppose** two graphs, X and Y, are plotted on the same set of axes and using the same scales. (You are **not** required to draw these graphs.) If X is a graph of the industrial production of the textile industry, relative to 1957, for the period 1958–63, and Y is a similar graph for the metal industry over the same period, then the number of intersections of the two graphs would be ④

A 0
B 1
C 2
D 3
E impossible to tell from the information given.

54 ③

x	$\frac{1}{2}$	1	2	3
y	$\frac{3}{4}$	3	12	27

that is when $x = \frac{1}{2}, y = \frac{3}{4}$; etc.

The above values of x and y are related by one of the following types of formula (k is a constant):

Type I $y = kx$
Type II $xy = k$
Type III $y = kx^2$

Select the appropriate formula, and having determined which it is, find the value of k for this particular set of values of x and y.

Questions 55–57

The daily requirement of vitamins is not the same for every person. It depends on the weight, age and sex of the person and on his state of health. The figures given in the following table therefore apply to a fictitious person, and refer to only five of the vitamins.

Vitamin	Daily requirement in milligrams	Number of milligrams in a serving								
		Milk	Cheese	Celery	Water-melon	Banana	Wheat-germ	Tuna	Egg	Liver
A	5	0·3	0·8	0·3	0·6	0·2	..	..	0·3	2·6
B_1	1·1	0·1	0·02	0·04	0·05	0·04	0·5	0·05	0·05	0·2
B_2	1·5	0·5	0·4	0·1	0·05	0·05	0·2	0·1	0·15	3
C	100	3·0	..	8·0	6·0	10·0	..	..	..	..
Niacin	17	0·3	..	0·2	0·2	0·7	1·3	12	0·05	14

55 If as many as 10 serves of the foods listed could be given each day, how many, by themselves, could provide the daily requirements for all of the 5 vitamins listed? ③

A 0
B 1
C 2
D 3
E 4

56 On one day a single serving of each of the foods listed is eaten. This would provide ④

A the daily requirement for at least three of the listed vitamins.
B the daily requirement of just two of the listed vitamins.
C at least half of the daily requirements of all the listed vitamins.
D all the daily requirements of all the listed vitamins.

57 Which of the following could provide the daily requirement of at least three of the listed vitamins? There may be more than one answer. ③

A 3 serves of liver
B an egg and a serve of wheat-germ
C 3 serves each of celery and cheese
D 1 serve of liver and 3 serves of watermelon
E 3 serves each of banana, celery, and wheat-germ

Questions 58–60

Two parallel lines have no intersection; they separate a plane into 3 regions. If a third line is drawn to intersect the first 2, there are 2 intersections and 6 regions. If a fourth straight line is drawn to intersect all the preceding lines (with no more than 2 straight lines passing through a common point), there are 5 intersections and 10 regions. Further intersecting lines are added according to the same rule.

Putting this information in table form:

Number of straight lines	2	3	4	5
Number of intersections	0	2	5	?
Number of regions	3	6	10	?

Complete the following statements:

58 With five lines the number of intersections is? ③

59 With five lines the number of regions is? ③

60 With eight lines the number of intersections is? ⑤

Questions 61–63

A factory has a production capacity of 100 000 articles. Its production costs may be grouped into fixed costs and variable costs. *Fixed costs* are incurred regardless of the volume of production. *Variable costs* are proportional to the volume of production. That is, if the production is doubled, variable costs are doubled.
For full production (100 000 articles) the following costs are estimated.

Labour	£50 000
Materials	£40 000
Manufacturing fixed costs	£10 000
Manufacturing variable costs	£20 000
Administrative expense	£20 000
Selling expense	£20 000

Note that:

(i) labour and materials should be considered as variable costs;
(ii) the selling expense is 50 per cent variable, 50 per cent fixed;
(iii) administrative expense is fixed;
(iv) the only revenue is from the sale of the articles.

61 For full production, what is the value of the fixed costs? ③

A £10 000 **C** £40 000
B £30 000 **D** £160 000

62 For half production, what fraction of the total costs are variable costs? ④

A $\frac{1}{3}$ **C** $\frac{3}{5}$
B $\frac{4}{5}$ **D** $\frac{1}{4}$

63 If 80 000 articles are produced and then sold at £2 each, this would result in ⑤

A a profit of £40 000
B neither a profit nor a loss
C a loss of £20 000
D a profit of £24 000
E a profit of more than £40 000

Questions 64–66

You are sales manager for a fuel refinery, making three different grades of fuel, represented by P, Q, R.
The sulphur content and price for each grade is listed in the table.

Grade	P	Q	R
Percentage weight of sulphur	2.2	1.8	1.4
Price per tonne (£)	58	62	70

To vary the percentage weight of sulphur content you may mix two or three of the three basic grades.
Any fuel, either a pure grade or a mixture, is referred to as a sample.

Each statement below consists of two requirements stipulated by a customer.

You are to write

A if you can meet neither requirement X nor requirement Y.
B if you can meet requirement X and requirement Y taken separately but not together.
C if you can meet requirement X and requirement Y taken together with just one sample.
D if you can meet requirement X and requirement Y taken together with more than one sample.
E if a case occurs which is not covered by A, B, C, and D.

64 Requirement X—a fuel of not more than 1 per cent sulphur ③
Requirement Y—a fuel costing exactly £76 per tonne

65 Requirement X—a fuel of not more than 2 per cent sulphur ④
Requirement Y—a fuel costing not more than £60 per tonne

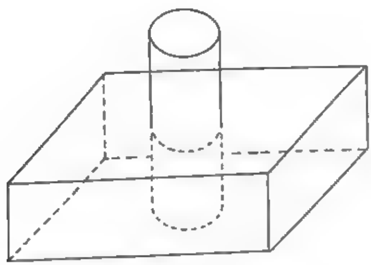
66 Requirement X—a fuel of exactly 2 per cent sulphur ④
Requirement Y—a fuel costing exactly £62 per tonne

Questions 67, 68 refer to the following information:

When a cylinder floats vertically in a liquid, as shown in the diagram, a certain fraction (f) of the block is in the liquid. If d_B is the density of the block and d_L is the density of the liquid, then it is known that

$$f = \frac{d_B}{d_L}$$

The density of any object is calculated by dividing its mass by its volume.



Below is a list of facts about various liquids, and a cylinder:

density of water	$= 1\,000$ kg m^{-3}
density of kerosene	$= 845$ kg m^{-3}
volume of cylinder	$= 0.25$ m^3
area of cross-section of cylinder	$= 1.25$ m^2
height of cylinder	$= 0.2$ m
mass of cylinder	$= 175$ kg

67 Of the six facts given in the list, what is the **least** number needed to determine the fraction of the cylinder which is above the liquid level when the cylinder floats in kerosene? ④

A 1

B 2

C 3

D 4

68 What fraction of the cylinder is below the surface when the cylinder floats in water? ④

A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{7}{10}$

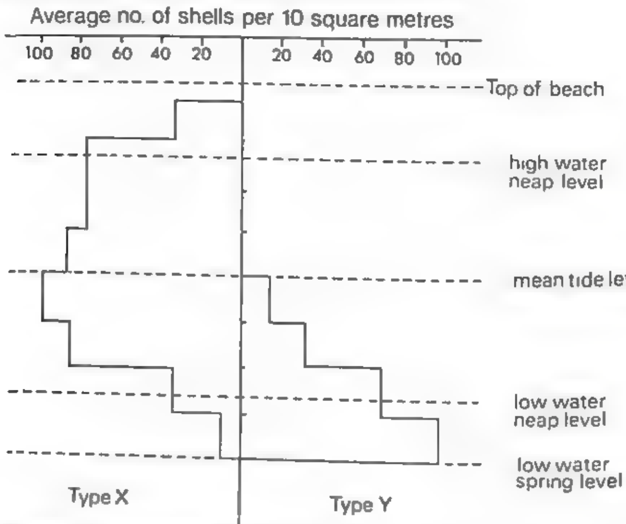
D $\frac{3}{4}$

E $\frac{4}{5}$

C GRAPHICAL

Questions 69–71

The following diagram relates to Questions 69–71.



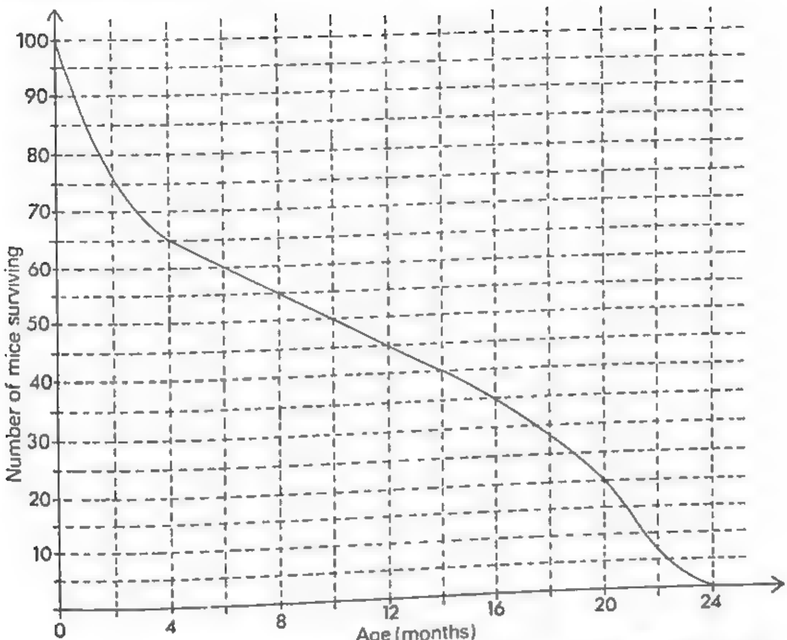
The diagram represents the results of an experiment designed to determine the distribution of two types of shells (X and Y) on a beach. The distribution of shells was only measured to low water spring level. The distribution shown above can be taken as representative of the distribution over the whole beach.
Use the information given above to answer the following questions.

69 Type X shells are ①
- **A** probably not found in quantity below low water spring level.
- **B** more plentiful at low water levels than at high water levels.
- **C** found on the beach in about the same numbers as type Y shells.
- **D** evenly distributed over the whole beach.

70 Type Y shells ①
- **A** are distributed over the whole beach.
- **B** are found higher up the beach than type X shells.
- **C** do not appear to be found above mean tide level.
- **D** do not occur below low water spring level.

71 There will be more type Y shells than type X shells on the beach ①
- **A** between mean tide and low water neap levels.
- **B** at mean tide level.
- **C** between low water and high water neap levels.
- **D** at low water neap level.

Questions 72–75



An experiment was conducted on the life span of white mice. 100 newly-born mice were isolated and not allowed to breed. The smoothed graph above shows the number surviving (vertical axis) at various ages (horizontal axis). Use the graph to answer Questions 72–75.

72 Which one of the following is the closest to the time that elapsed from the beginning of the experiment before exactly half of the original population had died? ①

A 2 months
B 6 months
C 10 months
D 14 months
E 18 months

73 Which one of the following is the best approximation to the number of mice that died within the first two months? ③

A 75
B 60
C 35
D 30
E 25
F 20

74 Which one of the following is the best approximation to the percentage of the original mice that died between 6 months and 12 months after the start of the experiment? ①

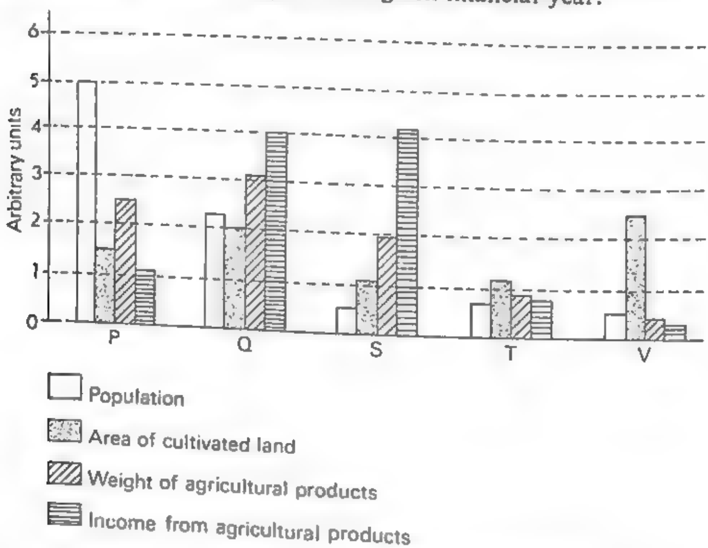
A 5 per cent
B 15 per cent
C 20 per cent
D 30 per cent
E 45 per cent
F 50 per cent

75 Consider those mice that survived for the first 6 months. Which one of the following is the closest to the time that elapsed from the beginning of the experiment before exactly half of these had died? ②

A 6 months
B 12 months
C 18 months
D 24 months
E 30 months

Questions 76–79

The column graph below shows the population, area of cultivated land, weight of agricultural products, and income from agricultural products for 5 country regions in a given financial year.



76 If the population of region Q is approximately 50 000, which one of the following is the best approximation for the total population of the five areas? ①

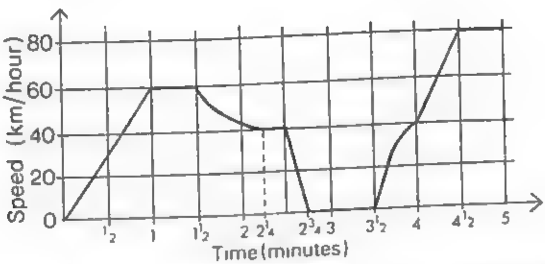
A 100 000
B 250 000
C 500 000
D 1 million
E 5 million

77 Which one of the regions P, Q, S, T, V has the greatest *total* land area? Write N if you think that there is not sufficient information in the graph to answer the question. ②

78 For which one of the regions P, Q, S, T, V will the income from agricultural products per cultivated hectare be greatest? Write N if you think that there is not sufficient information in the graph to answer the question. ①

79 For which one of the regions P, Q, S, T, V is the average income from each tonne of agricultural products the least? Write N if you think that there is not sufficient information in the graph to answer the question. ④

Questions 80–82



The above graph shows how the speed of a man's car varies over the first 5 minutes of his journey to work in the city. A speed limit of 60 km/hour applies throughout the journey. It is possible to obtain from the graph the distance the man has actually travelled, but for the purposes of this set of questions let this distance be x km.

80 For how long was the car stationary during the part of the journey shown in the graph? ①

A ½ minute
B ¾ minute
C 1 minute
D 1½ minutes
E 2 minutes

81 For how long did the man actually exceed the speed limit for the part of the journey shown? ①

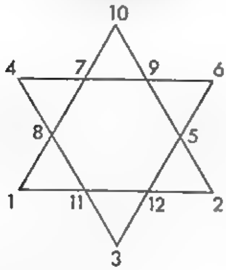
A ¼ minute
B ½ minute
C ¾ minute
D 1 minute
E none of these

82 If the man had been able to maintain a constant speed of 50 km/hour for the 5 minutes, he would have travelled ⑤

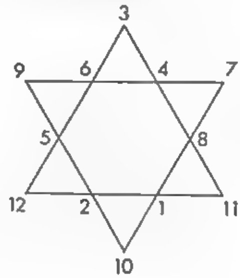
A less than x km
B x km
C further than x km
D either A, B or C, depending on the value of x.

90 The numbers along each straight line of the magic hexagram X add up to 26. To develop from X the magic hexagram Y, each number x on X is replaced by $(n + 1 - x)$. 2

Hexagram X



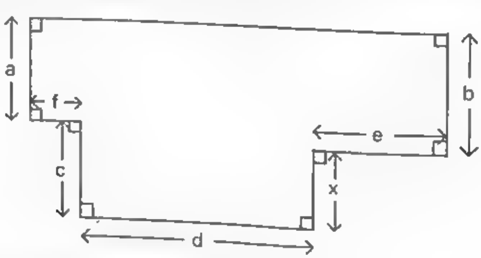
Hexagram Y



The value of n is

A 0
B 2
C 6
D 9
E 12
F 13

Questions 91–93 refer to the diagram, which has been drawn to scale and represents an irregular block of land.



91 The shortest straight section of the perimeter is of length 1

A a
B b
C c
D d
E e
F f
G x

92 Which one of the following correctly expresses x in as few of the other symbols as possible? 2

A $ac - b$
B $d - c$
C $b - (a + c)$
D $c + a - b$
E none of these

93 Which one of the following expressions does **not** correctly express the area of the block? ⑤

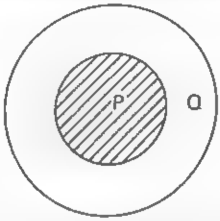
A $b(d + e + f) + xd + f(b - a)$

B $a(d + e + f) + (b - a)(d + e) + dx$

C $(a + c)(d + e + f) - fc - ex$

D $a(d + e + f) + cd + e(b - a)$

Questions 94, 95 refer to the following diagram and information:



Disc P lies on disc Q. The diameter of disc Q is twice the diameter of disc P. The diameter of disc P is x.

94 The area of disc P is ①

A x

B $2x$

C $3x$

D $4x$

E none of the above

95 The fraction of the area of disc Q *not* covered by disc P is ⑤

A $\frac{1}{3}$

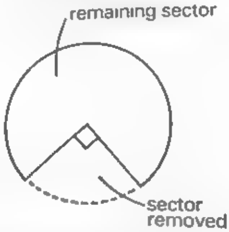
B $\frac{1}{4}$

C $\frac{3}{4}$

D $\frac{2}{3}$

E none of the above

96 A circular piece of paper has a sector of angle 90° cut from it. The two straight edges of the remaining sector are joined without overlap so that a cone is formed. ②



Which one of the following correctly gives the area of the curved surface of the cone as a fraction of the original area of the circle?

A $\frac{1}{4}$

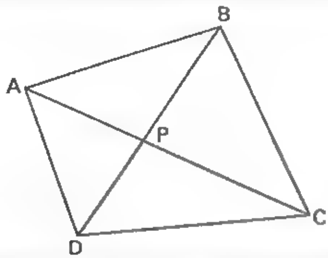
B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

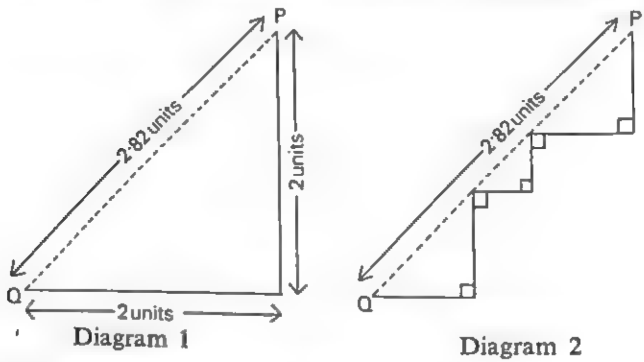
97 In the diagram, the diagonals AC and BD intersect at P. ②



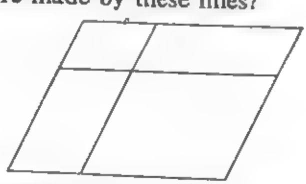
The number of triangles illustrated by this diagram is

A 4
B 6
C 8
D 10
E 12

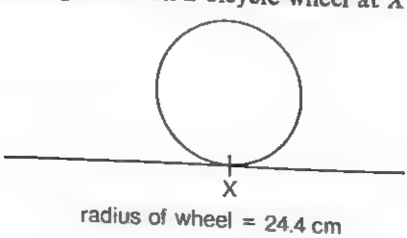
98 The solid lines in diagrams 1 and 2 show two different paths between P and Q. What is the length of the path in diagram 2? ②



99 This figure shows two sets of parallel lines. How many different parallelograms are made by these lines? ③



100 A white spot is painted on a bicycle wheel at X: ③



To a person sitting on the ground and carefully watching the white spot as the wheel is rolled along a level surface, it would appear to trace out the path as shown.

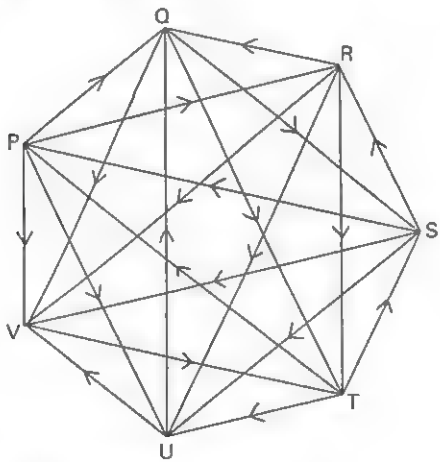
not drawn to scale

To the nearest whole number the value of x is

A 12
B 13
C 24
D 25
E 49

Questions 101–103

The diagram illustrates the decisions made by a man when considering the relative merits of seven different television sets, represented at P, Q, R, S, T, U and V. The man made his decisions by considering the sets in pairs.



The diagram is interpreted as follows:
TV set P was considered superior to Q.
TV set P was considered superior to R.
TV set T was considered superior to P, and so on.
Use this diagram to answer Questions 101-3.

101 Which one of the following sets was considered superior to set R? ②

A set U
B set T
C set V
D set Q
E set S

102 Which one of the following logical arguments is **not** supported by the diagram? ②

Argument A:

set P is considered superior to set U
set T is considered superior to set P
∴ set T is considered superior to set U

Argument B:

set P is considered superior to set Q
set S is considered superior to set P
∴ set S is considered superior to set Q

Argument C:

set U is considered superior to set V
set R is considered superior to set U
∴ set R is considered superior to set V

Argument D:

set V is considered superior to set T
set R is considered superior to set V
∴ set R is considered superior to set T

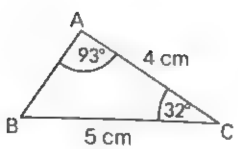
103 Suppose you depend entirely on the man's judgment, as represented in the diagram, when purchasing a TV set. Which one of the following sets would you purchase? ④

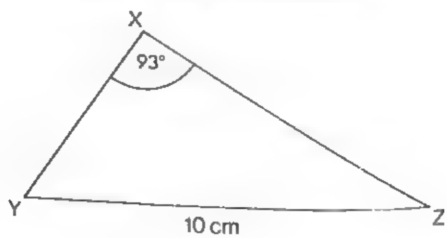
A set P
B set Q
C set R
D set S
E set U

Questions 104–106

Two triangles have dimensions as shown. For each of the following statements write

F if the statement is contradicted by the information given in the diagrams;
T if the statement is true for the information given in the diagrams;
C if the information given in the diagrams is not sufficient to determine whether the statement is true or otherwise.





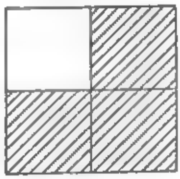
104 The length of XY is less than the length of YZ ②

105 $AB = 3$ cm ④

106 Angle $XYZ = 55°$ ④

Questions 107, 108

From the 60 matches in a box a boy builds a pattern thus: he forms a square with 4 matches (call this layer 1) ; to this he adds 3 more equal squares, as shown shaded in the diagram, to form layer 2 ; he continues to add layers in the same manner, so that after each layer is added the pattern is always a complete square ; the boy stops when he has added as many complete layers as possible with the 60 matches.



107 How many complete layers will the pattern have? ④

A 2 **D** 5

B 3 **E** 6

C 4 **F** more than 6

108 How many matches will be left over? ③

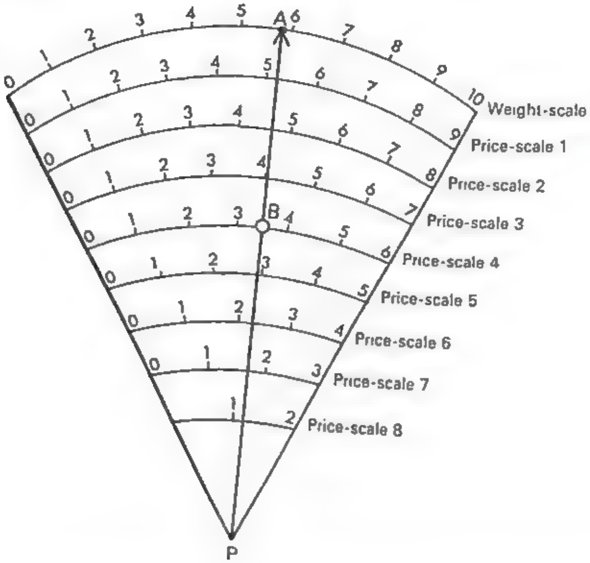
A 0 **D** 3

B 1 **E** 4

C 2 **F** more than 4

Questions 109–112

The diagram shows the face of a fruiterer's automatic weighing machine with the pointer pivoted at P. The top scale is marked in kilograms (weight-scale) and the scales under it are marked in pounds sterling (price-scales). £1 = 100 pence. Provided that he knows the price per kilogram of any fruit, the fruiterer can use the scales to read off, either without any calculation or with only simple addition, the price of any fruit he weighs.



For example, tomatoes cost 60p a kg. For the pointer in the position shown the fruiterer can say that he is weighing approximately 5.8 kg of tomatoes (point *A*) and the cost is £3.50 (point *B*).

109 Apples are sold at 30p a kg. To read directly from his weighing machine, without calculation, the cost of 7.3 kg of apples, the fruiterer will need to use ③

A price-scale 1.
B price-scale 4.
C price-scale 5.
D price-scale 7.
E one of the other price-scales.

110 The fruiterer is using his scales to find the cost of 4.6 kg of grapes, which are sold at 70p a kg. On the relevant price-scale the pointer will be between ③

A 1 and 2.
B 2 and 3.
C 3 and 4.
D 4 and 5.
E 5 and 6.

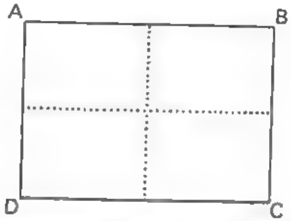
111 What is the cheapest fruit which the fruiterer can sell in order to be able to use his weighing machine to read off from one of the price-scales, without calculation, the price of a particular fruit? ④

A fruit costing 1 penny a kg
B fruit costing 2 pence a kg
C fruit costing 10 pence a kg
D fruit costing 20 pence a kg
E fruit costing other than that indicated in A, B, C, or D

112 If a special fruit costs £1.30 a kg, which one of the following pairs of price-scales could be used together to give the cost of this fruit? ④

A price-scales 3 and 7
B price-scales 2 and 5
C price-scales 2 and 6
D price-scales 1 and 8
E price-scales 3 and 6

Questions 113, 114



ABCD is a rectangular section of wall into which four tiles fit, as shown by the dotted lines.

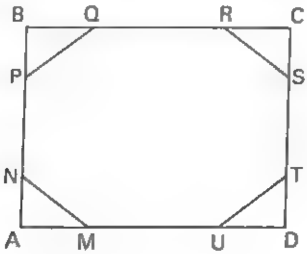
113 You are given two white tiles and two black tiles with which to fill the rectangle. How many different patterns can you make from these four tiles? ④

A 3
B 4
C 5
D 6
E 8
F 12
G 16

114 If you are given four tiles, each of which is white on one side and black on the other, how many different patterns can you make? ④

A 4
B 6
C 8
D 10
E 12
F 14
G 16

115 *ABCD* is a rectangle and M, N, P, Q, R, S, T, U are points on the sides such that $AM = \frac{1}{4}AD$, $AN = \frac{1}{4}AB$, $PB = \frac{1}{4}AB$, and so on. ④



The area of the octagon *MNPQRSTU* (on page 99) expressed as a fraction of the area *ABCD* is

A $\frac{3}{4}$

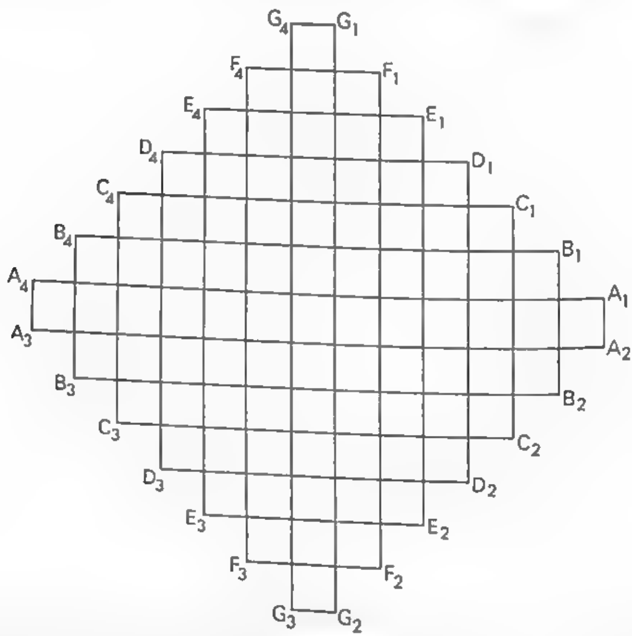
B $\frac{4}{5}$

C $\frac{7}{8}$

D $\frac{15}{16}$

E $\frac{31}{32}$

116 The diagram below is made up of a number of small squares each of side 1 unit. Seven rectangles $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, etc., have been indicated on the diagram. Let the perimeters of these be a units, b units, etc., respectively, and let their areas be A square units, B square units, etc., respectively. For the rectangle $A_1A_2A_3A_4$ the ratio $\frac{A}{a}$ is $\frac{13}{28}$. This ratio will vary from rectangle to rectangle. ④



Select the rectangle for which the ratio is the greatest. What is the value of the ratio in this case?

A $\frac{13}{28}$

B $\frac{33}{28}$

C $\frac{45}{28}$

D $\frac{49}{28}$

E $\frac{52}{28}$

F $\frac{85}{28}$

117 In mathematics we often represent numbers by points on a line in such a way that the distances between points representing consecutive whole numbers are equal. In the diagram below ④

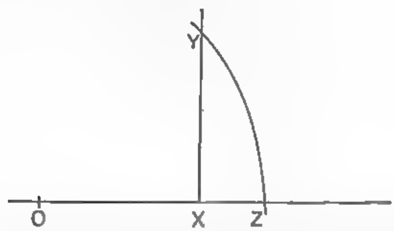
O represents the number zero

X represents the number 1

$OX = XY$

angle $OXY = 90°$

YZ is the arc of a circle whose centre is O.



Which one of the following numbers does Z represent?

A 1·5

B $\sqrt{2}$

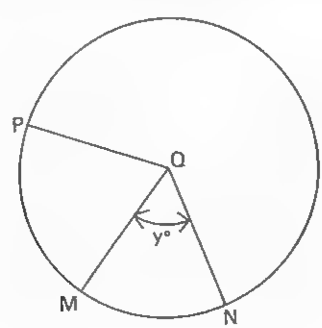
C $1\frac{1}{3}$

D $\frac{\pi}{2}$

E none of these

Questions 118, 119

The figure represents a speedometer graduated for speeds from 0 km/hour at M to 100 km/hour at N. (Other graduations are not shown.) The needle PQ is pivoted at Q, the centre of the circle, and its end P indicates speed on the graduated scale. The needle cannot move into the angle marked '$y°$'. The speed graduations are spaced evenly around the arc of the circle from M to N.



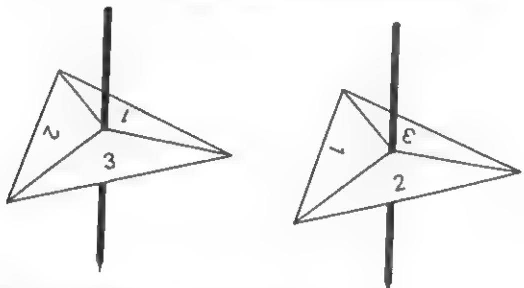
118 When the speed indicated is 40 km/hour then the angle PQM is equal to ④

A $(180 - y)°$
B $0{\cdot}4(360 - y)°$
C $(144 - y)°$
D $\frac{5}{2}(360 - y)°$
E none of these

119 If M, Q, and P are in the same straight line when the speed is 60 km/hour, find the value of y. ⑤

Questions 120, 121

A game is played with two numbered wheels. A turn consists of spinning both to give a score. The wheels are divided into three identical isosceles triangles numbered 1 to 3. For each wheel the score is the number on the face of the triangle whose base comes to rest on the table. Scores on the two wheels can be either added or multiplied to give the score for each turn. To get a progressive total score, the scores from each turn are added.



The winner of the game is the first person with a progressive total score which equals or exceeds 100. Two players, X and Y, play a large number of games. X obtains his score for each turn by addition and Y obtains his score for each turn by multiplication.

120 Which one of the following statements is true? ⑤

A X will probably win more games than Y.
B X and Y will probably win about the same number of games each.
C Y will probably win many more games than X.
D Insufficient information is given to make statements A, B, or C.

121 Which one of the following is the most likely number of turns player X will have to complete a game? ④

A 4
B 20
C 25
D 36
E 100

122 A thirteen-cm ruler, with four marks between the ends, allows you to measure, in one step, any integral length from 1 to 13 cm. For example, marks 0 and 1 can be used to measure 1 cm, 0 and 2 to measure 2 cm, 10 and 13 to measure 3 cm, and so on. For a 9-cm ruler, three marks between the ends are sufficient to measure, in one step, any integral length from 1 to 9 cm. Which one of the following sets of marks does **not** satisfy this requirement on the 9-cm ruler? ⑤

A marks 1, 2, and 6 cm from the left-hand end
B marks 1, 4, and 7 cm from the left-hand end
C marks 2, 5, and 8 cm from the left-hand end
D marks 3, 7, and 8 cm from the left-hand end
E marks 3, 4, and 7 cm from the left-hand end

Questions 123, 124

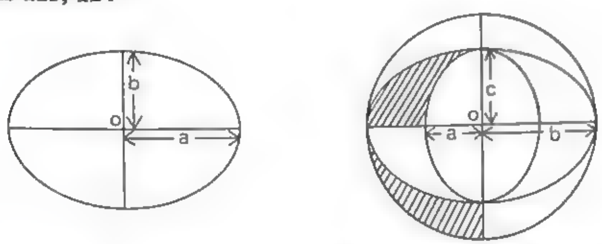


Figure 1 shows an ellipse centred at *o* Its area is πab square units.

Figure 2 shows two ellipses and a circle, all centred at *o*.

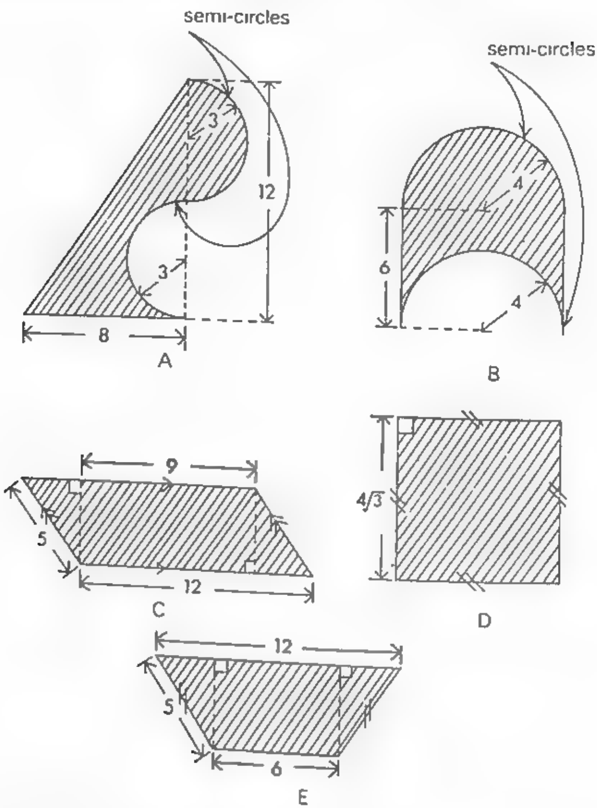
123 In figure 2 if the area of the smaller ellipse is to be half the area of the larger ellipse, then ⑤

A $b = 4a$
B $b = 2a$
C $a = 4b$
D $a = 2b$
E none of A, B, C, D is true

124 In figure 2 the total area of the shaded parts, in square units, is ⑤

A $\frac{\pi}{4}(b^2 - ac)$
B $\frac{\pi}{4}(bc - ac)$
C $\pi(b^2 - bc - ac)$
D $\pi(b^2 - ac)$
E $\frac{\pi}{2}(b^2 - bc - ac)$

125 Four of the shaded figures (over) have the same area. If all dimensions are in the same units, write down the letter of the figure which differs in area from the others. ⑤



126 If the radius of the complete circle is R cm and if the lines AB and CD are tangential to the arcs shown, which one of the following gives the measure of the shaded area of the diagram in square centimetres? ⑤

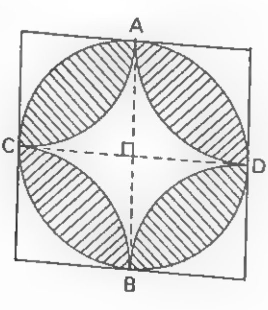
A $\frac{1}{2}\pi R^2$

B $\frac{5}{8}\pi R^2$

C $4R^2 - \pi R^2$

D $2\pi R^2 - 4R^2$

E R^2



E SYMBOLIC

127 a, b, c, and d are numbers such that c is less than b and d is greater than a. In view of this which one of the following statements is **not** true? ②

A $a + b$ may equal $c + d$
B $a + d$ may equal $b + c$
C $a + c$ may equal $b + d$
D $a + b + c$ may equal d
E $a + c + d$ may equal b

128 In this question the symbols Θ, Ξ, Δ, all represent real numbers. ②

If $\dfrac{\Theta \times \Delta}{\Xi} = \dfrac{\Theta \times \Theta \times \Delta \times \Delta}{Y}$

then Y is equal to

A $\Xi \times \Xi$
B $\Xi \times \Xi \times \Xi$
C $\Xi \times \Delta \times \Delta$
D $\Xi \times \Theta \times \Theta$
E $\Xi \times \Theta \times \Delta$

129 The equations to circles of diameters 6 cm and 10 cm may be written ②

$\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$ and $\dfrac{x^2}{25} + \dfrac{y^2}{25} = 1$ respectively.

The diameter of the circle with equation $\dfrac{x^2}{64} + \dfrac{y^2}{64} = 1$ is

A 8 cm
B 12 cm
C 16 cm
D 20 cm
E 48 cm
F none of the above

130 Another way of writing $(a + b)^2$ is $a^2 + 2ab + b^2$ which is, of course, the same as $b^2 + 2ba + a^2$. Similarly $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + \ldots + 5ab^4 + b^5$. The $\ldots$ represents a missing term. Which one of the following should replace the $\ldots$? ③

A $5ab$
B a^2b^3
C $15a^2b^4$
D $10a^2b^3$
E none of the above

Questions 131, 132 refer to the following information:

It may easily be proved that $\dfrac{1}{x} + \dfrac{1}{x} = \dfrac{1}{\frac{x}{2}}$

Use this relationship to answer Questions 131 and 132.

131 Which one of the following is an expression equivalent to ②

$\frac{1}{8\cdot4} + \frac{1}{8\cdot4}$?

A $\frac{1}{4\cdot2}$

B $\frac{1}{16\cdot8}$

C $\frac{2}{16\cdot8}$

D none of the above

132 Which of the following is an expression equivalent to ④

$\frac{1}{12\cdot8} + \frac{1}{12\cdot8} + \frac{1}{6\cdot4} + \frac{1}{1\cdot6} + \frac{1}{3\cdot2}$?

A $\frac{1}{0\cdot8}$

B $\frac{5}{36\cdot8}$

C $\frac{15\cdot2}{12\cdot8}$

D none of the above

Questions 133–135

Examine carefully the algebraic expressions below. In each case select the statement from the key which is most appropriate to the expression.

KEY

A true for all values of x
B false for all values of x
C false for all negative values of x
D false for all positive values of x
E false when x lies in the range 0 to 1 inclusive

133 $x < x + 2$ ②

134 $x > 2x$ ⑤

135 $x^2 > x$ ⑤

136 The cost of painting a cube on all faces is £2 per square metre if only one coat of paint is applied. It costs £10 to give a cube of side length x m one coat of paint. What is the maximum number of complete cubes each of side length $\frac{x}{2}$ m that can be painted for £10 if only one coat of paint is given to each? ⑤

A 2
B 4
C 5
D 8
E 16
F 20

F NUMERICAL

137 36 km/hour equals 10 metres/second. If an aircraft travels at 990 km/hour, what is its *approximate* speed in metres/second? ①

A 100 m/sec
B 200 m/sec
C 300 m/sec
D 400 m/sec
E 500 m/sec

138 During 1972 a man's savings increased by 10 per cent; during 1973 they decreased by 10 per cent. Over this two-year period what has happened to the value of the savings? ②

A It has increased.
B It has remained the same.
C It has decreased.
D It is impossible to tell without further information.

139 $0.75y$ m is 0.125 of the length of a room. It follows that the length of the room is ③

A $600y$ m
B $9.375y$ m
C $0.875y$ m
D $0.09375y$ m
E none of these.

140 Any odd number may be expressed as the sum of the two whole numbers nearest to its half. For instance ③

$$7 = 4 + 3 \qquad\qquad 9 = 5 + 4$$

If the odd number is called 'n' the general formula to express it as the sum of the whole numbers nearest to its half is:

A $n = \left(\frac{n}{2} + \frac{3}{2}\right) + \left(\frac{n}{2} - \frac{3}{2}\right)$

B $n = \left(\frac{n}{2} + 1\right) + \left(\frac{n}{2} - 1\right)$

C $n = \left(\frac{n}{2} + \frac{1}{2}\right) + \left(\frac{n}{2} - \frac{1}{2}\right)$

D $n = \left(\frac{n}{2} + \frac{1}{4}\right) + \left(\frac{n}{2} - \frac{1}{4}\right)$

141 Two projectiles speed directly towards each other, one at 12 000 km/hour and the other at 18 600 km/hour. They start 2 465 km apart. How far apart will they be 1 minute before collision? ④

A 1 946 km
B 110 km
C 225 km
D 510 km

142 Which of the following can be divided without remainder by at least three of the numerals 11, 3, 5, 2? There is more than one answer. ④

A 6 776
B 495
C 4 620
D 9 955 033
E 1 818
F 4 920

Questions 143–145

A positive integer is *prime* if it has only two factors, itself and one. (1 is not generally included as a prime.) A *prime number pair* consists of two prime numbers which differ by two.

143 How many prime numbers lie between 10 and 40? ④

A 4
B 7
C 9
D 10
E 15
F none of the above

144 How many prime number pairs lie between 10 and 40? ⑤

A 2
B 3
C 4
D 7
E 15
F none of the above

145 197 and 199 form a prime number pair. When 90 is added to each of them or subtracted from each of them, a prime number pair results ⑤

A in both cases.
B in neither case.
C from the addition but not from the subtraction.
D from the subtraction but not from the addition.

146 $12\frac{1}{2}$ per cent of a man's gross income is exempt from taxation. The ratio of his gross income to his taxable income is ⑤

A $\frac{8}{7}$
B $\frac{7}{8}$
C $\frac{8}{9}$
D $\frac{9}{8}$
E $\frac{1}{8}$

II ANALYSIS IN AN UNSTRUCTURED SYSTEM

A VERBAL

147 In a certain school, which offers only physics, chemistry and biology in its science courses, there are twelve teachers of science. Two of these teach biology but neither teaches physics nor chemistry. Eight teach physics and six teach chemistry. How many teach both physics and chemistry? ③

148 A certain fraction has a denominator which is 2 less than its numerator, x. By what fraction must this fraction be multiplied if their product is equal to one? ③

149 The side of a square sheet of cardboard is known to be an exact number of centimetres in length. Which is the only one of the following that could possibly represent its area? ③

A 441 square centimetres
B 635 square centimetres
C 1 000 square centimetres
D 1 728 square centimetres
E 8 000 square centimetres

Questions 150–152

A grandfather clock strikes 4 notes on the quarter hour, 8 on the half hour, and so on. Thus, on the hour it strikes 16 notes, and in addition it strikes the hours as well. In the house with the clock there are three visitors who, in the morning, make statements about the number of times they heard the clock strike during the night. Their statements are numbered as Questions 150–152. Circle on your answer sheet the letter

K if the visitor could be speaking the truth.
L if the visitor could not be speaking the truth.
M if there is no way of deciding whether the visitor's statement is true or false.

150 'I was awake for no more than half an hour, and the clock struck 20 times.' ④

151 'I awoke to hear the clock begin to strike, and it struck 15 times.' ②

152 'The clock woke me up striking continuously 33 times.' ①

153 A party of 7 adults and 7 children travels on a bus journey for which each adult pays the full fare of £1. A child if accompanied by its father travels free, otherwise it pays half fare. The total fare paid by the party is £9. On the return journey the party travels by train for which the fares are the same as on the bus except that a child must be accompanied by its mother in order to travel free. If the total fare for the return journey is £10, how many children in the party are not accompanied by a parent? ④

A none
B 2
C 3
D 4
E There is insufficient information supplied to answer the question.

Questions 154, 155

This year, Easter Day fell on March 26, which is early; but it is not as early as possible, for unless world-wide agreement to the contrary occurs, Easter Day is and will be the first Sunday after

the full moon which happens upon or next after the twenty-first day of March. And if the full moon happens upon a Sunday, Easter Day is the Sunday after.

Now the average time from one full moon to the next is 29 days, 12 hours and about 44 minutes, but the time may vary by as much as about 6 hours.

154 When is the earliest that Easter Day can occur? ④

A March 21
B March 22
C March 23
D March 24
E March 25

155 What is the latest day in April on which Easter Day could fall? ⑤

A thirtieth
B Anzac Day (twenty-fifth)
C first
D the Sunday following the second Saturday
E none of these

Each of the Questions 156 to 158 consists of a problem followed by an answer. Write

A if the information given in the problem is **not sufficient** to determine whether the answer is correct ;
B if the information is **just sufficient** to determine the answer and shows that the answer given is reasonable ;
C if the information is **just sufficient** to determine the answer and shows that the answer given is obviously incorrect ;
D if the information is **more than is needed** to determine the answer but shows that the answer given is reasonable ;
E if the information is **more than is needed** to determine the answer and shows that the answer given is obviously incorrect.

156 A man bought 500 articles when they were selling at 65p each. A fortnight later he purchased a second lot of the articles, making the average cost of both transactions 66p per article. What was the price per article of the second lot of articles purchased? ④
ANSWER—67p

157 A gramophone record which rotates at the rate of $33\frac{1}{3}$ revolutions per minute takes eighteen minutes to play. If the diameter of the inside groove is 12 cm and that of the outside groove 30 cm, find the average distance between successive grooves. ⑤
ANSWER—0.015 cm

158 Paper of uniform width (30 cm) and thickness (0.0125 cm) is rolled on to a spool of diameter 5 cm. What length of paper has been rolled on when the diameter of the roll is 30 cm? ⑤

ANSWER—30 m

159 The outside area of a circular cylinder closed at one end is given by $A = \pi R(R + 2h)$. If R and h are each made three times as great as previously how many times greater will the area become? ④

A 3

B 6

C 9

D 18

E 27

160 A cube has edge of length x cm and a circle has radius r cm. Express x in terms of r if the fractions ⑤

$$\frac{\text{volume of cube}}{\text{surface area of cube}} \quad \text{and} \quad \frac{\text{area of circle}}{\text{circumference of circle}} \quad \text{are equal.}$$

Write your answer in its simplest form.

161 If y is a multiple of 5 what is the next number greater than $(y + 2)$ which is exactly divisible by 5? ④

162 The numerators are missing in three of the following fractions. What is the result when the sum of the first two fractions is divided by the sum of the last two? ⑤

$$\frac{2x}{y} - \frac{\quad}{3y} = \frac{\quad}{y^2} = \frac{\quad}{ay}$$

163 The difference between the value of the 5 and the value of the 7 in the number 57 is 43. The number n is composed of two digits, a being the tens digit and b the units digit. Write down an expression for the difference in value between the digits of the number n. ⑤

164 Consider the problem: 'Mary is twice as old as her sister Joan is now. How old will Joan be when Mary is twenty?' ⑤
If you consider that there is a single answer to the problem in its present form write Y. If you consider that more information is required, study the statements below and select every one which would, by itself, provide sufficient additional information for a solution to be obtained.

- **A** Mary's present age
- **B** Joan's present age
- **C** the sum of Mary's and Joan's present ages
- **D** the difference between Mary's and Joan's present ages
- **E** the time in years before Mary will be twenty

B TABULAR

165 In a pile of beads 30 per cent are red and the remainder blue. There are 360 beads in the pile. A second pile contains 300 beads. How many in the second pile are red if half of the two piles combined are red? ③

- **A** 20
- **B** 60
- **C** 70
- **D** 210
- **E** 222
- **F** 330

Questions 166–168 relate to the following information and table:

In order to decide which of two cars to purchase, a man obtained the following information:

Information	Car X	Car Y
A Purchasing price	£1 250	£1 450
B Running costs (including registration, insurance, petrol, oil, etc.)	4p/km	5p/km
C Petrol consumption	12 km/l	10 km/l
D Depreciation (based on value at beginning of each year)	by $\frac{1}{4}$ of its value annually	by $\frac{1}{6}$ of its value annually
E Average annual kilometrage	16 000 km	16 000 km

NB—Depreciation indicates the loss in value of the car as it gets older.

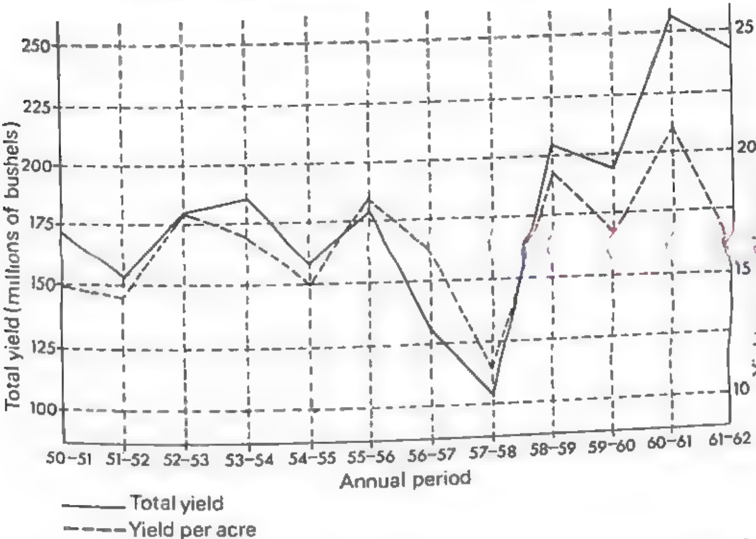
166 The man intends to keep the car for three years. Which one of the above pieces of information would not be of use in calculating the amount saved, during the three years, by purchasing one car in preference to the other? ⑤

167 What would be the difference (to the nearest £) in the annual cost of petrol for the two cars if petrol were 13p per litre? ⑤

168 What would be the value of car Y at the end of two years? ⑤

C GRAPHICAL

Questions 169–171 refer to the following graphs based on the production of wheat in Australia for the period 1950–51 to 1961–62:



One graph shows the total yield of wheat for each annual period. The other graph shows the average yield of wheat per acre sown for each annual period.

169 The average number of acres of wheat sown per year in Australia is approximately ④

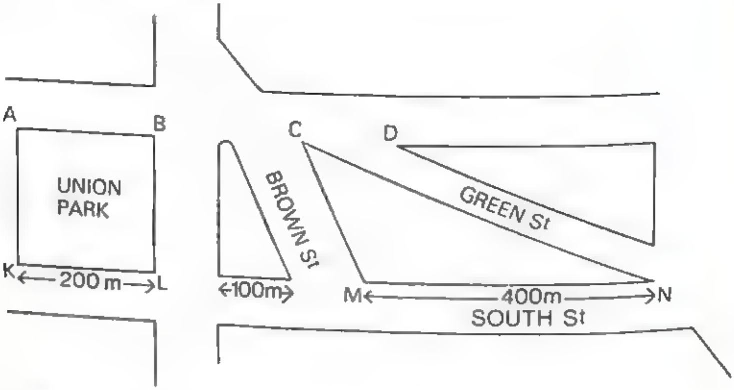
A 100 000 acres.
B 500 000 acres.
C 1 000 000 acres.
D 10 000 000 acres.
E 150 000 000 acres.

170 For which annual period was the number of acres of wheat sown the least? ⑤

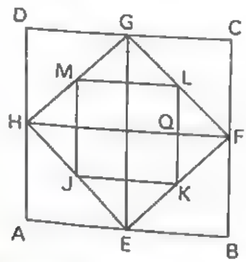
171 For which annual period was the number of acres of wheat sown the greatest? ⑤

D FIGURAL

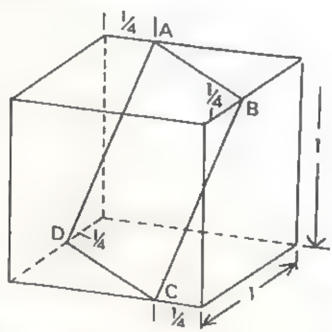
172 In the diagram Union Park is a rectangular garden of area 40 000 square metres. (The points A, B, C, and D are in a straight line and so are the points K, L, M, and N.) What is the area of the block bounded by Green, Brown, and South Streets? ②



173 In the diagram $ABCD$ is a square and E, F, G, H, J, K, L, M are mid-points of sides as shown. What fraction of the area of $ABCD$ is the area of QKF? ②



Questions 174 and 175 refer to a plane quadrilateral which is fitted into a unit cube as shown:



174 It is possible to infer, without calculation, that ②

A $AB = BC$

B $AB = CD$

C $AB < BC$

D $AB < CD$

E $AB > BC$

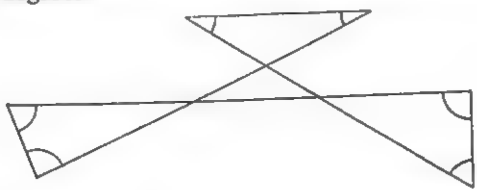
F $AB > CD$

175 The length of AB is ③

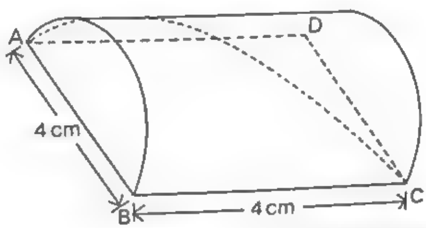
A $\sqrt{\frac{9}{8}}$ **C** $\sqrt{\frac{9}{2}}$

B $\sqrt{\frac{1}{2}}$ **D** $\sqrt{3}$

176 In the diagram find the sum of the marked angles. Give your answer in degrees. ③



177 The figure shows half a circular cylinder of length 4 cm and diameter 4 cm. ④

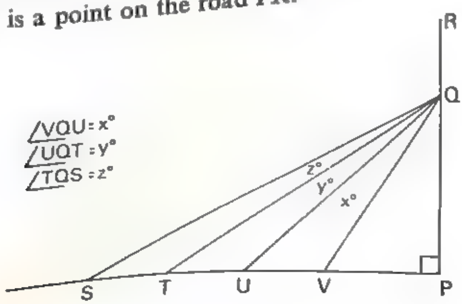


What is the shortest distance from A to C along the curved surface?

A 8 cm
B $4\sqrt{2}$ cm
C $2\pi + 2$ cm
D $4(\pi + 1)$ cm
E $\sqrt{4\pi^2 + 16}$ cm

Questions 178, 179

A number of lamp-posts S, T, U, V are equally spaced along a straight road SP. A second straight road PR meets SP at right angles. Q is a point on the road PR.



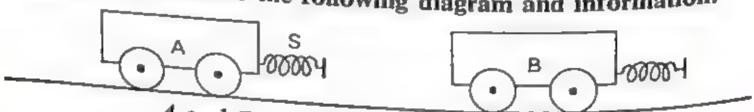
178 Which one of the following statements is true? ④

A $z - y = y - x$
B $x = y$
C $z < x$
D $2y = x + z$
E $y > x$

179 If Q represents the position of a car travelling at constant speed towards P and along the road RP, then ④

A the size of angle QVP decreases more rapidly than the size of angle QUP.
B the size of angle QVP decreases at the same rate as the size of angle QUP.
C the size of angle QVP decreases more slowly than the size of angle QUP.
D any one of A, B, and C may be true, depending on the speed of the car.

Questions 180, 181 refer to the following diagram and information:



A and *B* represent identical carts

Cart *A* rolls along a table at a speed of 12 centimetre per second. Cart *B* is initially at rest. As soon as the end of spring *S* (projecting in front of *A*) touches *B*, *B* starts to move. *B* moves in all a total distance of 6 centimetre from its initial position before the spring no longer touches it. At the instant this occurs Cart *A* has become stationary, and remains so, and Cart *B* now has a speed of 12 centimetre per second.

180 The total time during which the spring contacted Cart *B* was 2 seconds. At what time after the initial contact was the spring probably most compressed? ④

A 0.5 second
B 1 second
C 1.5 second
D 2 second
E at all times during the contact

181 The carts had the same speed ④

A when cart *B* was 3 centimetre from its initial position.
B at some point when cart *B* was less than 3 centimetre from its initial position.
C at some point when cart *B* was more than 3 centimetre from its initial position.
D at no stage during the collision.

E SYMBOLIC

182 If n is a positive whole number greater than one, select the statement concerning the number $(n - 1) \times n \times (n + 1)$ that is always true. $(n - 1) \times n \times (n + 1)$ is always ④

A exactly divisible by $2n$
B greater than 6
C exactly divisible by 3
D greater than n^3
E not less than n^4

183 $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ is equal to one of the following. Select it. ⑤

A $(x - y)^4$
B $x^4 - y^4$
C $(x + y)^4$
D $x^4 + y^4$
E $(2x - 2y)^4$

III READING AND COMPREHENSION OF NEW MATHEMATICAL MATERIAL (DEFINITIONS—SYSTEMS)

A VERBAL

184 In the study of logic in elementary mathematics a proposition is defined as a sentence to which only one of the terms true or false can be meaningfully applied. Select from the following the sentence which is **not** a proposition. ⑤

A This triangle is an equilateral triangle.
B All squares are parallelograms.
C $x^2 + 4x + 4 = (x + 2)(x + 2)$
D $9 = 7 + 4$
E $2x + 3 = 12$

B TABULAR

Questions 185–187

Consider a new kind of multiplication, restricted to the symbols 0, 1, 2, 3, and for which the multiplication table is shown below. If we take the first factor from the top row, and the second factor from the left-hand column, then the result of multiplication is found in the cell that is common to that particular row and column. Thus $3 \times 1 = 3$, because the number in the last column and the second row is 3, as underlined. Similarly $2 \times 3 = 2$ because the number in the second to last column and last row is 2, as underlined.

When a and b are used in Questions 185-187 they represent the symbols 0, 1, 2, 3.

×	0	1	2	3
0	0	0	0	0
1	0	1	2	$\underline{3}$
2	0	2	0	2
3	0	3	$\underline{2}$	1

185 The relationship $a \times b = b \times a$ ②

- **A** is true only if $a = b$
- **B** is true only if $a = 0$ or $b = 0$
- **C** is true for all possible values of a and b
- **D** is not true for any value of a or b

186 If a is not equal to 0 for what value of a will $a^2 = 0$? ④

187 Find the value of 3^3 using the above method of multiplying. ④

C GRAPHICAL

Questions 188–191

Number pairs which are written (a, b) can be used in a way similar to ordinary single numbers. For example, number pairs can be added in the following way:

$$(a, b) + (c, d) = (a + c, b + d)$$

And number pairs can be multiplied in the following way:

$$(a, b) \bullet (c, d) = ac + bd$$

188 $(3, -3) \bullet (-4, 4)$ equals ①

- **A** -10
- **B** 20
- **C** 30
- **D** -24
- **E** none of these

189 $(3, 4) + (2, 6) - (1, 3)$ equals ②

- **A** (3, 5)
- **B** (4, 6)
- **C** (6, 13)
- **D** (4, 7)
- **E** none of these

Questions 190 and 191 refer to the following diagrams and information.

The number pair (x, y) can be represented as a point on a graph, as shown in Figure 1.

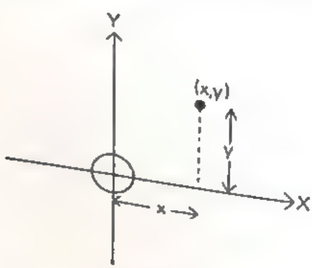


Figure 1

Figure 2 shows such a plane in which the points A, B, C, D, E are marked.

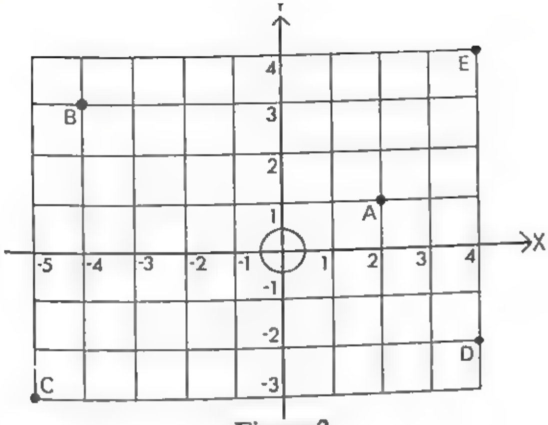


Figure 2

190 Which point represents $(-5, 3) + (1, 0)$? ①

191 Which point represents $(2, 3) + (-7, -6)$? ①

Questions 192–195 refer to the following diagrams and information:

Ordinary numbers can be represented as points along a line $0X$, as shown in Figure 1. The number line is always drawn in this direction. x is the symbol for any number on this line.

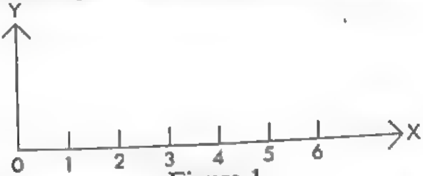


Figure 1

When the letter K is placed in front of a number, this means that the number line has been rotated about 0, anti-clockwise through 90°. So the number $K5$ would be located at point M in Figure 2, as the number line has rotated anti-clockwise through 90° and is now in the direction of $0Y$.

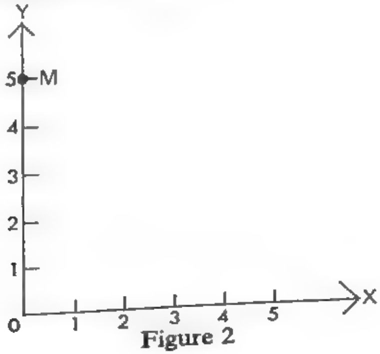


Figure 2

Figure 3 shows a plane space in which points (or number pairs) may be located relative to the axes $0X$ and $0Y$.

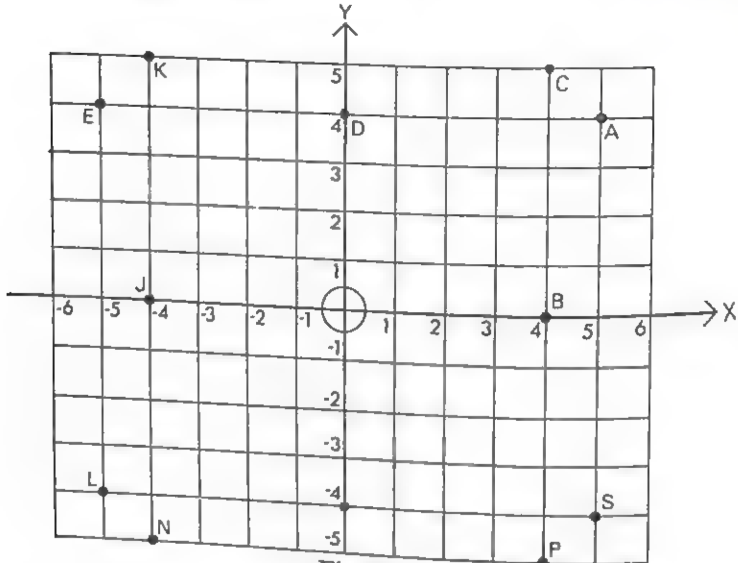


Figure 3

Name (by its appropriate letter) the point on this space which could represent the following numbers:

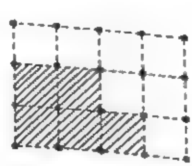
192 $K4$ ①

193 $K(K4)$ ③

194 $-K4$ ①

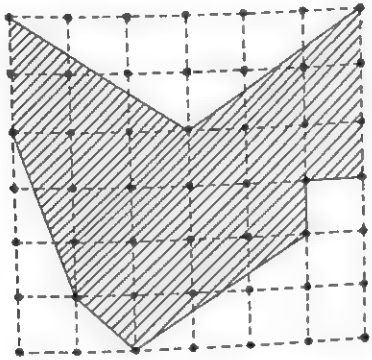
195 $K5 + 4$ ①

196 The grid shown is called a lattice and contains 20 lattice points. The shaded area, if every square is counted as a square unit, is 5. This area may be expressed by the lattice points in the following way: ②



area (in square units)
$= \frac{1}{2}$ (sum of all lattice points on the perimeter) + sum of all lattice points inside the perimeter − 1
$= \frac{1}{2}(10) + 1 - 1$
$= 5$

Whenever the vertices of a plane figure with straight sides are all lattice points this method may be used to calculate the area of the figure. Apply the method to find the area shaded in the diagram below. The area shaded is:



A 13 square units.
B 16 square units.
C 17 square units.
D 20 square units.
E 21 square units.
F 36 square units.

Questions 197–199

In algebra $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is given a specific meaning. It has the value $ad - bc$. For example,

$$\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = 4 \times 6 - 3 \times 5 = 9$$

②

197 The value of $\begin{vmatrix} 7 & 3 \\ -5 & 2 \end{vmatrix}$ is

A −29
B −1
C 11
D 29
E 31

198 Which one of the following will leave the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ unaltered? ②

A change a with d and then change b with c

B change a with b and then change c with d

C change a with d and then change d with b

D change a with b and then change b with c

E change a with c and then change b with d

199 If each letter in $\begin{vmatrix} \mathrm{p} & \mathrm{q} \\ \mathrm{r} & \mathrm{s} \end{vmatrix}$ is multiplied by 3, then its value is ④

A unaltered

B multiplied by 3

C multiplied by 6

D multiplied by 9

E multiplied by 12

D FIGURAL

Questions 200–203 refer to the following diagram and information:

Figure 1 (on the next page) shows a simplified flow chart for a computer. The flow chart consists of a series of steps. Each step is boxed separately. In other words, this is a list of instructions for the computer.

N refers to the number of cases.

X is the numerical value of each case.

Σ means 'the sum of'.

ΣX means 'the sum of all the numerical values of X already processed'.

M is the mean (average) numerical value of N cases.

K is the number of cases already processed.

The cases fed into the computer have the following numerical values, and are read by the computer in this order:
2, 7, 5, 7, 9, 12, 5, 9.

200 What is the value of N? ③

A 2
B 3
C 7
D 8
E 56

201 What is the value of M? ②

A 1
B 3
C 7
D 8
E 9

202 If K = 3, what is the numerical value of the datum (X) which the computer will read next? ④

A 3
B 4
C 5
D 7
E 9

203 One of the instruction boxes reads ③

$$\Sigma X = \Sigma X + X.$$

This instruction

A means X must equal zero.
B means add the value of the next case to the existing total.
C means the computer should add all values of X together.
D asks the computer whether all cases have been counted.

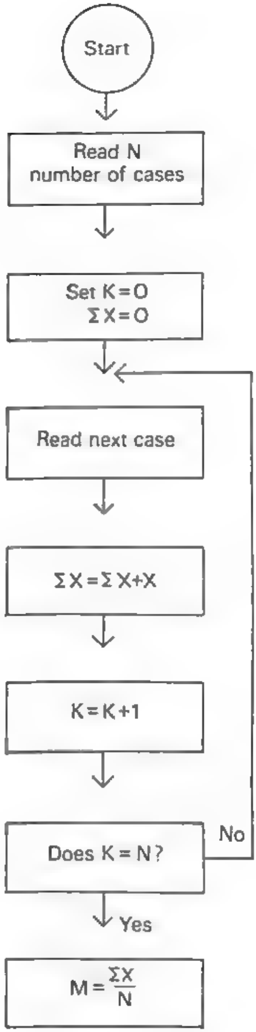


Figure 1

E SYMBOLIC

Questions 204–208 refer to the following information:

The expression ($a \triangle b$) means the product of the number a and the number ($b + 1$); a or b may have any value including zero.

Thus $3 \triangle 4 = 15$.

For Questions 204 and 205 write

A if the statement is always true.
B if the statement is sometimes true.
C if the statement is never true.
D if there is insufficient information to decide the truth or falsity of the statement.

204 $0 \triangle b = 0$, where 0 means zero. ②

205 $1 \triangle b = b$ ③

206 What is the value of $(-4) \triangle (-3)$? ②

A 6 **C** 15
B 8 **D** 16

207 If $3 \triangle x = -3$, then ③

A $x = -1$
B $x = -3$
C $x = 0$
D the value of x can be calculated, but it is not listed above.
E there is no value of x which fits.

208 If $a \triangle b$ is equal to 0 and $a \neq 0$, then ④

A $b = 0$
B $b = 1$
C $b = -1$
D no conclusion can be drawn about the value of b.

209 A cyclic expression in three symbols, p, q, and r, is one in which replacement of p by q, q by r, and r by p, all at the same time, gives an expression which is the same as the original expression. For example, $3(p + q + r) - p^2q^2r^2$ is cyclic because when we replace p by q, q by r, and r by p, we have $3(q + r + p) - q^2r^2p^2$ and this is equal to the original expression. ③

Which one of the following expressions is **not** cyclic?

A $p^3 + q^3 + r^3 - pqr$
B $(p - q)(q - r)(r - p)$
C $p^2q + p^2r + q^2r$
D $(p + q + r)^2$
E $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$

Questions 210–212

In algebra the symbol $n!$ is given a specific meaning. It stands for the product

$$n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$

For example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Use this information to answer Questions 210–212.

210 Find the numerical value of $(3\,!)^{2!}$ ③

211 Find the numerical value of $\dfrac{11\,!}{9!}$ ④

212 $\dfrac{(2n)\,!}{n\,!}$ is always equal to ⑤

A $2\,!$
B $2n$
C 2
D n
E $n\,!$
F none of the above

213 An algebraic expression in x and y is said to be symmetrical when its value is unaltered by interchanging the x and the y. Select from the following the one which is **not** symmetrical. ⑤

A $x^2 + y^2$
B $3xy$
C $(x + y)^2$
D $(x - y)^2$
E $x^2 - y^2$

214 As an abbreviation for '$3 \times 2 \times 1$' we write $3\,!$ Similarly '$5 \times 4 \times 3 \times 2 \times 1$' is represented by $5\,!$ If $\dfrac{x\,!}{(x-1)\,!} = 8$ what is the value of $\left(\dfrac{x}{2}\right)!$? ⑤

A 4/7
B 4
C 6
D 12
E 24
F none of these

215 In this question the symbol $[x]$ means 'the largest integer which is less than or equal to x'. For example, ⑤

$$[4\tfrac{1}{7}] = 4, \qquad [\tfrac{22}{4}] = 5, \qquad [13\tfrac{1}{3}] = 13.$$

Which one of the following statements is correct for all values of x greater than 1?

A $x[x] = x^2$
B $\dfrac{x}{[x]} = 1$
C $[x + 1] = [x] + 1$
D $[x(x - 1)] = [x][x - 1]$

Questions 216–218

A different method of writing fractions is to write them as number pairs. For example, the fraction $\frac{2}{3}$ may be written as (2, 3). The operations applicable to fractions apply as usual. Using this system of arithmetic, answer Questions 216–218.

216 The value of (1, 2) + (1, 4) is ①

A (1, 2)
B (2, 6)
C (2, 4)
D (3, 4)
E (3, 8)

217 The value of (3, 5) ÷ (2, 3) is ①

A (2, 5)
B (6, 15)
C (9, 10)
D (15, 6)
E (10, 9)

218 The value of $\frac{(5, 7) \times (14, 15)}{(2, 3)}$ is ①

A (1, 1)
B (4, 9)
C (0, 1)
D (9, 4)
E (1, 0)

Questions 219–221 refer to the following information:

The following list gives some examples of the use of star numbers:

$$*2 + *3 = *6$$
$$*4 + *5 = *20$$
$$*8 + *2 = *16$$

219 Complete the following: $*3 + *4 =$ ①

A $*7$
B 7
C $*12$
D 12

220 Complete the following: $*12 - *6 =$ ④

A $*6$
B 6
C $*2$
D 2
E −72
F $*{-72}$

221 Complete the following: $*\left(\frac{p}{q}\right) =$ ④

A $*p + *q$
B $*p - *q$
C $*p \div *q$
D $*p \times *q$

Questions 222–225

In this set of questions $a \wedge b$ means 'the greatest common factor of a and b' and $a \vee b$ means 'the lowest common multiple of a and b'. For example $12 \wedge 8 = 4$ and $12 \vee 8 = 24$.

222 $(5 \vee 10) \times (3 \wedge 6)$ is equal to ③

A 15
B 20
C 30
D 40
E 60
F 150

223 If $a \vee b = c$ then the value of $2a \vee 2b$ is ②

A c **D** c^2
B $2c$ **E** $4c^2$
C $4c$

224 If a and b are even numbers then ②

A both $a \wedge b$ and $a \vee b$ must be even.
B only $a \wedge b$ must be even.
C only $a \vee b$ must be even
D neither $a \wedge b$ nor $a \vee b$ must be even.

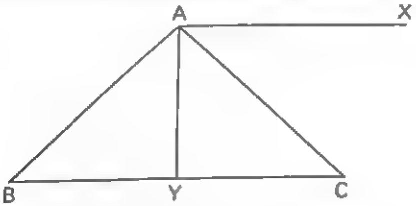
225 If $a \wedge b = k$ then the value of $a^2 \wedge b^2$ is ③

A k **D** k^2
B $2k$ **E** k^4
C $4k$

IV FOLLOW AND CONSTRUCT A PROOF

D FIGURAL

226 ABC is any triangle and AX is drawn parallel to BC. AY is drawn perpendicular to BC. In the following proof that $\angle YAB = \angle YAC$, which statement is the first that is not justified by the information? ②



To prove: $\angle YAB = \angle YAC$

Proof:

$$\angle YBA + \angle YAB = 90^\circ \text{ (statement K)}$$
$$\angle XAC + \angle YAC + 90^\circ = 180^\circ \text{ (statement L)}$$
$$\angle XAC + \angle YAC = 90^\circ \text{ (statement M)}$$
but $$\angle XAC = \angle YBA \text{ (statement N)}$$
$$\therefore \angle YBA + \angle YAC = 90^\circ \text{ (statement P)}$$
$$\angle YAB = \angle YAC$$

227 A rectangular sheet of paper, corners $ABCD$, is cut into two parts by cutting along the diagonal BD. A point, P, is chosen on BD such that $BP = \frac{3}{4}BA$, and a further cut is made along the straight line PA. ⑤
$AB = 12$ cm and $AD = 5$ cm.
Consider the following statements in the light of this information. If you think that the reasoning is sound throughout, circle the letter E on your answer sheet. If you think that the reasoning is not correct, circle the letter corresponding to the first step in which an error occurs.

	$BP = \frac{3}{4}BA$	(given)
	$\therefore\ BP = 9\,\text{cm}$	(statement A)
Since	$\angle BAD = 90°,\ BD = 13\,\text{cm}$	(statement B)
	$\therefore\ PD = 4\,\text{cm}$	(statement C)
But	$DA = 5\,\text{cm}$	(given)
	$\therefore\ PA = 3\,\text{cm}$	(statement D)

E SYMBOLIC

228 In an examination four candidates score marks a, b, c, d. The average for the four is denoted by x. Later a fifth candidate sits for the examination and his mark e goes with the other four marks to make a new average y such that $y = 2x$. Now study the method of finding the relationship between e and x. ③

Step A	$a + b + c + d = 4x$
Step B	$a + b + c + d + e = 5y$
Step C	$a + b + c + d + e = 10x$
Step D	$4x + e = 10x$
Step E	$e = 6x$

Write S if you think that the working is correct. If you think that the working is not correct, write the letter corresponding to the first step in which an error occurs.

229 In a certain experiment the numbers a, b, c, d are such that ④

$$a > b > c > d.$$

Now study this reasoning:

	$(a + b) > (c + d)$
Step F	$\therefore\ a > (c + d - b)$
Step G	$\therefore\ -a > (-c - d + b)$
Step H	$\therefore\ (d - a) > (b - c)$
Step I	$\therefore\ (d + c) > (b + a)$.

If you think that the reasoning is sound throughout circle the letter J, on your answer sheet. If you think that the reasoning is not correct, circle the letter corresponding to the first step in which an error occurs.

Questions 230–233

The following consists of the statement of a theorem, notes about the terms used and a proof of the theorem. Two lines (line X and line Y) have been omitted. Study the material given and then answer the questions which follow it.

Theorem $\sqrt{2}$ is not a rational number.

Notes $\sqrt{2}$ stands for the number which gives 2 when multiplied by itself.

A rational number is one which can be expressed as the fraction p/q, where p, q are whole numbers ($q \neq 0$) and have no factor in common, e.g., $\frac{5}{7}, \frac{2}{1}$.

Proof If $\sqrt{2}$ was rational it could be written as p/q where p and q are whole numbers ($q \neq 0$) and have no common factor i.e., $\sqrt{2} = p/q$

Squaring and rearranging leads to

....(*line X*)

Thus p^2 is even and so p must be even.

Let $p = 2r$

Thus $2q^2 = 4r^2$

or $\quad q^2 = 2r^2$

Hence q must also be even.

....(*line Y*)

But this contradicts the original assumption in the proof. Therefore $\sqrt{2}$ is not rational.

230 From the following select the number which is not rational. ③

A $\sqrt{0.81}$

B $\sqrt{2\frac{1}{4}}$

C $\sqrt{16}$

D $\sqrt{5}$

E $\sqrt{25/16}$

231 Which one of the following would be the most suitable to write in *line X*? ③

A $2 = p/q$

B $q^2 = 2p^2$

C $2q = p$

D $p^2 = 2q^2$

E $q^2p^2 = 2$

232 Which one of the following statements would be the most suitable to write in *line Y*? ⑤

A Hence p and q cannot be rational.

B Hence p/q is not rational.

C Hence $p = q = 2r$.

D Hence p and q have the factor 2 in common.

E Hence p and q are both perfect squares.

233 Which one of the following best summarizes the method used to establish the truth of the statement '$\sqrt{2}$ is not a rational number'? ⑤

A Assume that the statement is false and then show that this leads to a contradiction.

B Assume that the statement is true and then show that this cannot be directly supported by logical means.

C Deduce logically from a known fact that the statement cannot be contradictory.

D Develop logically a contradiction of the statement from a known fact.

234 Study the following sequence of steps. It is possible that in some of the steps the left-hand side of the statement does not equal the right-hand side. Give the letter corresponding to the first line on which this occurs. If there is no such line write N. ⑤

Given $x = 3$

line **A** $\therefore (x - 3) = 0$

line **B** $\therefore (x - 3)\, x = (x - 3)\, 4$

line **C** $\therefore x^2 - 3x = 4x - 12$

line **D** $\therefore x^2 - 7x + 12 = 0$

line **E** $\therefore (x - 3)\,(x - 4) = 0$

line **F** $\therefore x - 4 = 0$

line **G** $\therefore x = 4$

line **H** $\therefore$ as $x = 3$ and $x = 4$, therefore $3 = 4$

ANSWERS TO APPENDIX II

1 E
2 T
3 R
4 M
5 L
6 D
7 B
8 A
9 B
10 4
11 1 km
12 B
13 A
14 E
15 A
16 B
17 F
18 B
19 E
20 B
21 E
22 C
23 C
24 B
25 D
26 C
27 E
28 B
29 A, D
30 B
31 B
32 A
33 E
34 B
35 C
36 A
37 D
38 B
39 E
40 C
41 J
42 A
43 E
44 B
45 D
46 A
47 E
48 B
49 D
50 A
51 C
52 C
53 C
54 Type III, k = 3
55 A
56 A
57 A
58 9
59 15
60 27
61 C
62 C
63 D
64 A
65 C
66 B
67 C
68 C
69 A
70 C
71 D
72 C
73 E
74 B
75 C
76 B
77 N
78 S
79 P
80 B
81 C
82 C
83 D
84 D
85 C
86 A
87 F
88 B
89 B
90 E
91 F
92 D
93 A
94 E
95 C
96 E
97 C
98 4
99 9
100 E
101 E
102 B
103 D
104 T
105 F
106 C
107 D
108 A
109 D
110 C
111 B
112 B
113 D
114 9
115 C
116 D
117 B
118 B
119 60
120 B
121 C
122 E
123 B
124 A
125 E
126 D
127 C
128 E
129 C
130 D
131 A
132 A

133	A
134	D
135	E
136	B
137	C
138	B
139	E
140	C
141	D
142	B, C, F
143	F
144	B
145	D
146	A
147	4
148	$\frac{x-2}{x}$
149	A
150	K
151	L
152	L
153	E
154	B
155	E
156	A
157	B
158	E
159	C
160	$x = 3r$
161	$y + 5$
162	1
163	$10a - b$
164	A, B, C, D, E
165	E
166	C
167	£35
168	£128
169	D
170	56–57
171	61–62
172	40 000 m^2
173	$\frac{1}{32}$
174	B
175	A
176	360°
177	E
178	C
179	A
180	B
181	B
182	C
183	A
184	E
185	C
186	2
187	3
188	D
189	D
190	B
191	C
192	D
193	J
194	M $(0, -4)$
195	C
196	E
197	D
198	A
199	D
200	D
201	C
202	D
203	B
204	A
205	C
206	B
207	D
208	C
209	C
210	36
211	110
212	F
213	E
214	E
215	C
216	D
217	C
218	A
219	C
220	C
221	B
222	C
223	B
224	A
225	D
226	N
227	D
228	S
229	G
230	D
231	D
232	D
233	A
234	B

This book has grown out of several ye[illegible] experience by the author as a chief e[illegible] the Commonwealth Secondary Schoo[illegible] Examination.

The early chapters deal with diagnostic [illegible] achievement tests, mastery tests, tests of mathematical learning, predictive tests, generalizability, analysis of a test and how to produce an objective test.

Transcripts of two panel meetings of item-writers have been included, to enable readers to gain some insight into the tremendous importance of dialogue in the test-construction process.

65

Simple methods are given for determining the internal consistency of a test and for estimating its error.

Following this are over 70 pages of test items, with answers.

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